TRISECTING AN ANGLE

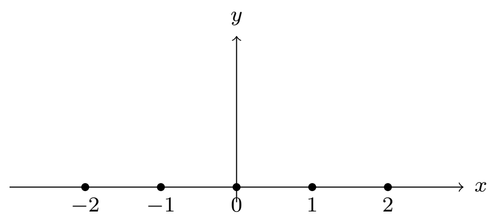
The ancient Greeks tried to find a method to trisect an arbitrary angle using just a straightedge and compass and failed. In this series of exercises you will show that they were doomed to fail! In particular, you will show that it is IMPOSSIBLE to trisect the angle $\frac{\pi}{3}$ using just a straightedge and compass. This shows that any proposed method to trisect an arbitrary angle with a straightedge and compass is doomed to failure!

Remarks.

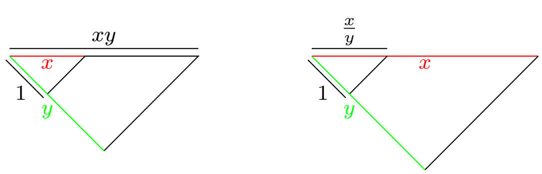
- Throughout these exercises "construction/construct" means "construction/construct with a straightedge and compass".
- The set of rational numbers (numbers like $\frac{1}{2}$, $-\frac{7}{4}$ etc.) is denoted by $\mathbb{Q}$.

Exercises.

(a) Fix a unit distance and show that you can construct coordinate axes and any integer on the x-axis as shown below:

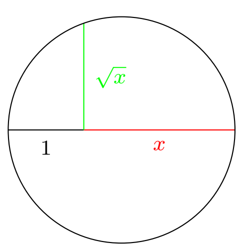


(b) Show that you can add, subtract, multiply, and divide lengths using a straightedge and compass (some pictures are show below to help):



Conclude that you can construct any rational number on the x-axis.

(c) Show that you can take square roots of lengths using a straightedge and compass (see the diagram below).



Conclude that you can construct any point in

$$\mathbb{Q}(\sqrt{t}) := \{r + s\sqrt{t} : r, s, t \in \mathbb{Q}\}.$$

(d) When we constructed the regular pentagon in the lecture, we started with $\mathbb{Q}$ and then we showed that the diagonal of a pentagon had length $\frac{1}{2} + \frac{\sqrt{5}}{2}$. We then constructed $\sqrt{5}$ so that we could construct any point in $\mathbb{Q}(\sqrt{5})$ which allowed us to construct the regular pentagon.

Convince yourself that when you construct an object that you progress through a series of stages where at each stage you can construct points on the x-axis of a certain type. For example, when we constructed the regular pentagon at stage 1 we could construct any rational number and at stage 2 we could construct any point from $\mathbb{Q}(\sqrt{5})$. We can represent these stages in the following way:

$$\begin{array}{ccc} \mathbb{Q} & \subseteq & \mathbb{Q}(\sqrt{5}) \\ \text{stage } 1 & & \text{stage } 2 \end{array}$$

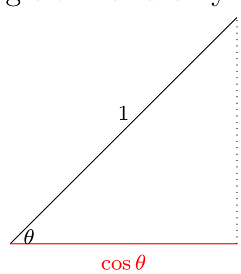
(e) (**tricky but important**) Denote the set of points that you can construct at stage k of a construction by S_k. Show that points in S_{n+1} are of the form $a + b\sqrt{c}$ where a, b, c are in S_n.

$$\begin{array}{ccccccccc} S_1 = \mathbb{Q} & \subseteq & S_2 & \subseteq & \dots & \subseteq & S_n & \subseteq & S_{n+1} \\ \text{stage } 1 & & \text{stage } 2 & & & & \text{stage } n & & \text{stage } n+1 \end{array}$$

(**Hint:** every point in S_{n+1} is constructed by intersecting lines and circles. Lines must pass through two points in S_n and a circle must have a centre in S_n and have a radius whose square is in S_n. See the diagram below).

Exercises (a)-(e) shows that every distance you can construct is of the form $a + b\sqrt{c}$ where a, b, c are also constructible.

(f) Show that you can construct an angle θ if and only if you can construct $\cos(\theta)$.



(g) Use the formulas

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

to show that

$$\cos(\tfrac{\pi}{3}) = 4\cos^3(\tfrac{\pi}{9}) - 3\cos(\tfrac{\pi}{9}). \tag{1}$$

(h) Use (1) to conclude that $2\cos(\frac{\pi}{9})$ is a root of the equation

$$f(x) = x^3 - 3x - 1. \tag{2}$$

Exercises (f)-(h) show that if you can trisect $\frac{\pi}{3}$ then you can construct a root of $f(x) = x^3 - 3x - 1$.

(i) Show that if $f(x)$ has a root of the form $c + d\sqrt{a}$ then $c - d\sqrt{a}$ is also a root of $f(x)$.

(j) If $\cos(\frac{\pi}{9})$ can be constructed then it can be done so in n stages:

$$\begin{array}{ccccccc} S_1 = \mathbb{Q} & \subseteq & S_2 & \subseteq & \dots & \subseteq & S_n \\ \text{stage } 1 & & \text{stage } 2 & & & & \text{stage } n \end{array}$$

If $\cos(\frac{\pi}{9}) = c + d\sqrt{a} \in S_n$ (with $c, d, a \in S_{n-1}$) is a root of $f(x)$ then $c - d\sqrt{a_n}$ is also a root of $f(x)$. Use this to show that $f(x)$ has a solution in S_{n-1}.

(k) By the previous exercise $f(x)$ has a root in S_{n-1}. Use a similar idea to the ones in the previous two exercises to show that $f(x)$ has a root in S_{n-2}.

(l) Repeat the argument in the previous exercise to show that $f(x)$ has a root in S_{n-3} and hence a root in S_{n-4} ... and hence a root in $S_1 = \mathbb{Q}$.

(m) Show that if $\frac{a}{b}$ is a root of $f(x)$ then both a and b divide 1, and conclude that $f(x)$ does not have a root in $\mathbb{Q}$.

Exercises (i)-(m) show that if $f(x) = 0$ has a constructible solution then $f(x) = 0$ has a solution in $\mathbb{Q}$. You have also shown that $f(x) = 0$ does not have a solution in $\mathbb{Q}$ and hence does not have a constructible solution.

CONGRATULATIONS! When you put the blue boxes together you have shown that it is not possible to trisect $\frac{\pi}{3}$ with a ruler and compass; namely, it is not possible trisect every angle!