MAS277 Exam Solutions
Spring Semester, 2012-2013
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1.

(i) **bookwork**
V is finite-dimensional if it has a finite spanning set. Alternatively, if it has a finite basis.

(ii) **bookwork**
The dimension of V is the number of elements in a basis. Every finite-dimensional vector space has a dimension because every vector space has a basis, and every basis has the same number of elements.

(iii) **seen similar**
(a) Assume that U is not finite dimensional, then it has no finite spanning set. Pick a nonzero $u_1 \in U$, and then a $u_2 \notin \mathbf{sp}\{u_1\}$ which must exist since the set $\{u_1\}$ cannot span by hypothesis. Then $\{u_1, u_2\}$ is linearly independent, and does not span, so pick an element $u_3 \notin \mathbf{sp}\{u_1, u_2\}$. The set $\{u_1, u_2, u_3\}$ is then also linearly independent, and we can continue this process to find linearly independent sets $\{u_1, \ldots, u_k\}$ in U for any finite k. The Steinitz Exchange Lemma states that any linearly independent subset must be smaller than a spanning set, so if we choose $k = \dim V + 1$ then we can find a linearly independent set with k elements, but by definition V has a spanning set with only $k - 1$ elements, a contradiction. Hence U must also be finite-dimensional.

(b) We have the formula

$$\dim U + \dim W = \dim (U \cap W) + \dim (U + W).$$

Since we know $\dim U + \dim W = \dim V$, we get

$$\dim V = \dim (U \cap W) + \dim (U + W).$$

Under these assumptions, we have that $U \cap W = 0$ if and only if $\dim (U \cap W) = 0$ if and only if $\dim V = \dim (U + W)$ if and only if $V = U + W$. Since $V = U \oplus W$ holds by definition when $U \cap W = 0, U + W = V$, we have shown that $U \cap W = 0$ if and only if $V = U \oplus W$.

(c) U consists of those matrices of the form

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

so is clearly of dimension 3. W consists of those matrices of the form

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

so is clearly of dimension 6. If $A \in U \cap W$, then A must have zeros along the main diagonal and below, and by anti-symmetry must also therefore have zeros above, so $A = 0$. Thus we see that

$$\dim U + \dim W = 3 + 6 = 9 = \dim M_3(\mathbb{R})$$

and $U \cap W = 0$, so we must have $M_3(\mathbb{R}) = U \oplus W$.

2.
(i) **bookwork**
Choose a basis $\mathcal{V} = \{v_1, \ldots, v_n\}$ for V and a basis $\mathcal{W} = \{w_1, \ldots, w_m\}$ for W. Then we can write

$$Fv_1 = \sum \alpha_{i1} w_i$$

in a unique way; do the same for each basis vector v_j:

$$Fv_j = \sum \alpha_{ij} w_i.$$

Then the matrix $[F_{\mathcal{V},\mathcal{W}}]$ for F with respect to these bases is $[\alpha_{ij}]$.
(ii) **seen similar**
Let $\mathcal{V}'$ be a second basis for V, and let A be the change-of-basis matrix from $\mathcal{V}$ to $\mathcal{V}'$. Then

$$[F_{\mathcal{V}',\mathcal{V}'}] = A[F_{\mathcal{V},\mathcal{V}}]A^{-1},$$

so the determinant on the left is equal to the determinant on the right. But the determinant on the right is equal to the product of determinants

$$(\det A)(\det [F_{\mathcal{V},\mathcal{V}}])(\det A^{-1}) = (\det A)(\det A^{-1})(\det [F_{\mathcal{V},\mathcal{V}}]) = \det [F_{\mathcal{V},\mathcal{V}}]$$

since $(\det A)(\det A^{-1}) = 1$.
(iii) **bookwork**
The kernel of F is

$$\{v \in V : F(v) = 0\}.$$

The image of F is

$$\{w \in W : w = F(v) \text{ for some } v \in V\}.$$

F is an isomorphism if it is injective and surjective, or equivalently if there exists a linear map $G : W \to V$ such that $FG = 1_W, GF = 1_V$. F is an isomorphism if and only if its kernel is 0 and its image is W.
(iv) **seen similar**
(a) To show that E is linear, we must check that $E(f+g) = E(f) + E(g)$ and that $E(\alpha f) = \alpha E(f)$.

$$\begin{aligned} E(f+g) &= (f+g)'' - 2(f+g)' + (f+g) \\ &= f'' + g'' - 2(f' + g') + f + g \\ &= f'' - 2f' + f + g'' - 2g' + g \\ &= E(f) + E(g) \end{aligned}$$

$$\begin{aligned} E(\alpha f) &= (\alpha f)'' - 2(\alpha f)' + (\alpha f) \\ &= \alpha f'' - 2\alpha f' + \alpha f \\ &= \alpha(f'' - 2f' + f) \\ &= \alpha E(f) \end{aligned}$$

(b) We compute the action of E on each of the basis vectors as below.

$$\begin{aligned} E(1) &= 1 \\ E(\sin x) &= -2\cos x \\ E(\cos x) &= 2\sin x \\ E(\sin 2x) &= -3\sin 2x - 4\cos 2x \\ E(\cos 2x) &= -3\cos 2x + 4\sin 2x \end{aligned}$$

Thus the matrix $[E]$ with respect to this basis is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 4 \\ 0 & 0 & 0 & -4 & -3 \end{bmatrix}$$

and so the determinant of E is $1 \cdot -2 \cdot -2 \cdot (9+16) = 100$.

3.
(i) **bookwork**
Assuming it exists, the adjoint of F is a linear map $G : W \to V$ such that

$$\langle Fx, y \rangle = \langle x, Gy \rangle$$

for all $x \in V, y \in W$.
(ii) **seen similar**
First, clearly $0 \in V$. If $f, g \in V$, then

$$(f+g)^{(n)} = f^{(n)} + g^{(n)}$$

so

$$(f+g)^{(n)}(0) = (f+g)^{(n)}(1) = 0$$

since we know $f, g \in V$, so $f + g \in V$. If $f \in V$ and $\alpha \in \mathbb{R}$, then

$$(\alpha f)^{(n)} = \alpha f^{(n)}$$

so $\alpha f \in V$ as well.

To show that D is a linear transformation from V to V, first note that $Df \in V$ if $f \in V$ as $(Df)^{(n)} = f^{(n+1)}$ so if $f^{(n)}(0) = f^{(n)}(1) = 0$ for all n then the same will hold for Df. If $f, g \in V$ and $\alpha \in \mathbb{R}$, then we have

$$D(f+g) = (f+g)' = f' + g' = Df + Dg,$$

$$D(\alpha f) = (\alpha f)' = \alpha f' = \alpha Df,$$

so $D : V \to V$ is a linear map.
(iii) **unseen**
To find the adjoint D^* of D, we have to find a linear map $D^* : V \to V$ such that

$$\langle Df, g \rangle = \langle f, D^* g \rangle$$

for all $f, g \in V$.

$$\begin{aligned} \langle Df, g \rangle &= \int_0^1 f'(x) g(x) \\ &= f(x)g(x)|_0^1 - \int_0^1 f(x) g'(x) \\ &= \int_0^1 f(x)\,(-g'(x)) \end{aligned}$$

The last equality holds since $f(0) = g(0) = f(1) = g(1) = 0$. Define $D^*(g) = -g'$; this is a linear map $V \to V$ by the same argument as for D, and the calculation above shows it is the adjoint of D.
(iv) **unseen**

Note that for D^2 to be self-adjoint we must have $\langle D^2 f, g\rangle = \langle f, D^2 g\rangle$ for all $f, g \in V$. Now $D^2 = D \circ D$ so we have the following calculation.

$$\begin{aligned}\langle D^2 f, g\rangle &= \langle D(Df), g\rangle \\ &= \langle Df, D^* g\rangle \\ &= \langle f, D^*(D^* g)\rangle \\ &= \langle f, D^2 g\rangle\end{aligned}$$

The final equality is $D^*(D^*g) = D^*(-g') = -(-g')' = g''$. (This can also be done directly using two applications of integration-by-parts.)

4.

(i) **unseen**

We will check that $\langle x, y\rangle_A$ gives an inner product on $\mathbb{R}^2$.

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$$\begin{aligned}\langle x + x', y\rangle_A &= (x + x')^T Ay \\ &= (x^T + x'^T)Ay \\ &= x^T At + x'^T Ay \\ &= \langle x, y\rangle_A + \langle x', y\rangle_A\end{aligned}$$
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$$\begin{aligned}\langle \alpha x, y\rangle_A &= (\alpha x)^T Ay \\ &= \alpha x^T Ay \\ &= \alpha\langle x, y\rangle_A\end{aligned}$$
- Viewing a number as a 1×1-matrix which is its own transpose, we have
$$\begin{aligned}\langle x, y\rangle_A &= \langle x, y\rangle_A^T \\ &= (x^T Ay)^T \\ &= y^T A^T x^{TT} \\ &= y^T Ax \\ &= \langle y, x\rangle_A\end{aligned}$$
since A is symmetric.
- If $x = [x_1\ x_2]^T$, then $\langle x, x\rangle_A$ is
$$x_1^2 - 8x_1x_2 + 20x_2^2 = (x_1 - 4x_2)^2 + 4x_2^2$$
which is greater than or equal to 0, and equals 0 only when $x_1 = x_2 = 0$.

(ii) **bookwork**

The norm $||v||$ is $\sqrt{\langle v, v\rangle_A}$. The norm of e_2 is

$$\sqrt{(0-4)^2 + 4} = \sqrt{20}.$$

(iii) **bookwork**

Take a basis $v_1, \ldots, v_n$, and define $\mathbf{proj}_u(v) = \frac{\langle u,v\rangle}{\langle u,u\rangle}u$. Define

$$\begin{aligned}u_1 &= v_1 \\ u_2 &= v_2 - \mathbf{proj}_{u_1}(v_2) \\ u_i &= v_i - \textstyle\sum_{j=1}^{i-1} \mathbf{proj}_{u_j}(v_i)\end{aligned} .$$

The orthonormal basis is $w_i = \frac{u_i}{||u_i||}$.

(iv) **seen similar**

Since $||e_2|| = \sqrt{20}$, the first vector $w_1 = [0\ \frac{1}{\sqrt{20}}]^T$. The vector u_2 is

$$u_2 = e_1 - \mathbf{proj}_{e_2}(e_1) = [1\ \ \frac{1}{5}]^T.$$

Now $[1\ \ \frac{1}{5}]^T$ has norm $\frac{1}{\sqrt{5}}$, so the second vector in this orthornormal basis is $\sqrt{5}[1\ \ \frac{1}{5}]^T$.