277 Revision Class 26/3

Exam: 4 Questions, all compulsory

Q n is based on chapter n material

25 marks per question ~ 8 for definitions & statements
~ 8 standard examples
~ 9 {short (un)seen proof examples

Testing as much material as possible. Don't leave anything out.

Chapter 1

* Fields & Vector spaces
* Subspaces $U+W$, $U \oplus W$, $U \cap W$

$U+W := \{\underline{v} \in V \mid \underline{v} = \underline{u} + \underline{w} \text{ for } \underline{u} \in U \text{ \& } \underline{w} \in W\}$

Show that $U+W$ is a subspace of V

To show that $U+W$ is a subspace of V we need to show:

a) $\underline{0} \in U+W$
b) $\underline{x} + c\underline{y} \in U+W$ if $\underline{x}, \underline{y} \in U+W$ & $c \in \mathbb{F}$

* linear (in)dependence
* bases & dimension

$U+W$ is written $U \oplus W$ when every $\underline{x} \in U+W$ is written as $\underline{x} = \underline{u} + \underline{w}$ for unique $\underline{u}$ & $\underline{w}$

Ex $U \cap W = \{0\} \Leftrightarrow U + W = U \oplus W$

Def 5 $U \subseteq V$ linearly dependent if $\exists$ $\underline{u}_1, \ldots, \underline{u}_n \in U$ & $c_1, \ldots, c_n \in \mathbb{F} \setminus \{0\}$ s.t. $c_1\underline{u}_1 + \ldots + c_n\underline{u}_n = \underline{0}$.

Ch 2

* linear transformations/maps
* invertible linear transformations, isomorphisms, $V \cong \mathbb{F}^n$
* Kernel, rank, rank-nullity theorem
* Coordinate vector relative to a basis
* Matrix representation of a linear map relative to bases
* Determinant, trace, eigenvalues & eigenvectors (tomorrow)

2005-2006 (PMA 220) Q1

a) Let V & W be vector spaces over $\mathbb{R}$. Define what it means for a map $\alpha : V \to W$ to be linear. (3 marks)

* (Good) $\alpha : V \to W$ is said to be linear if & only if:
  - $\alpha(\underline{u} + \underline{v}) = \alpha(\underline{u}) + \alpha(\underline{v})$ $\forall$ $\underline{u}, \underline{v} \in V$
  - $\alpha(c\underline{v}) = c\alpha(\underline{v})$ $\forall$ $\underline{v} \in V$ & $c \in \mathbb{R}$ $\square$

(Q: Show $L : V \to W$ is linear where $L(\underline{v}) = \ldots$

To show L is linear ~~&~~ it is

sufficient to show $L(\underline{u}+c\underline{v})=L(\underline{u})+cL(\underline{v})$
$\forall\ \underline{u},\underline{v}\in V$ & $c\in\mathbb{F}$)

* (Bad) α linear $\Rightarrow \underbrace{\alpha(\underline{u}+c\underline{v})=\alpha(\underline{u})+c\alpha(\underline{v})}$

What is $\underline{u},\underline{v},c$?

Should be $\Leftrightarrow$

Use this in practise, it is a contraction of the whole def.

α isn't stated to be a map.

b) Which of the following maps are linear? Justify your answers briefly, giving specific counterexamples where appropriate.
iii) $\chi : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by $A^TA = \chi(A)$

* (Good) Note that for $c=2$ & $A=I_{2\times2}$ we have

$$\chi\left(2\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)=\chi\left(\begin{pmatrix}2&0\\0&2\end{pmatrix}\right)$$

$$=\begin{pmatrix}2&0\\0&2\end{pmatrix}\begin{pmatrix}2&0\\0&2\end{pmatrix}=4\begin{pmatrix}1&0\\0&1\end{pmatrix}=2^2\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$=4\begin{pmatrix}1&0\\0&1\end{pmatrix}^T\begin{pmatrix}1&0\\0&1\end{pmatrix}=4\chi\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)$$

$$\neq 2\,\chi\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)$$

ie. $\chi(cA)$ does not equal $c\chi(A)$

$\forall c \in \mathbb{R}$ & $A \in M_{2\times 2}(\mathbb{R})$. Hence χ is not linear.

* (Bad) $\chi(A+cB) = (A+cB)^T(A+cB)$

$= (A^T + cB^T)(A+cB)$

$= A^TA + A^TcB + cB^TA + c^2B^TB$

$\neq A^TA + cB^TB = \chi(A) + c\chi(B)$

No counterexample given

To turn into a good answer, add "because when ... $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ & $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ & $c = 2$,

$\chi(A+cB) = 4B^TB \neq 2B^TB$

$= c\chi(B)$"

Steinitz Exchange Lemma

② a) Let V & W be finite-dimensional vector spaces & let $\alpha : V \to W$ be a linear map. Prove that there is a number $r \geq 0$ & lists $\underline{v}_1, \ldots, \underline{v}_n \in V$ & $\underline{w}_1, \ldots, \underline{w}_m \in W$ s.t.

i) $\underline{w}_1, \ldots, \underline{w}_m$ is a basis for W.

ii) $\alpha(\underline{v}_i) = \underline{w}_i$ for $1 \leq i \leq r$

iii) $\underline{v}_{r+1}, \ldots, \underline{v}_n$ is a basis for $\ker(\alpha)$

Prove also that $\underline{v}_1, \dots, \underline{v}_n$ is linearly independent.

Set $\dim(V) = n$ & $r = \dim(\mathrm{Im}(\varphi))$

The rank-nullity theorem tells us that

$$n = \dim(V) = \dim(\mathrm{Im}(\varphi)) + \dim(\ker(\varphi))$$

i.e. $\dim(\ker(\varphi)) = n - r$

Let $\{\underline{b}_{r+1}, \dots, \underline{b}_n\} = B$ be a basis for $\ker(\varphi)$. We extend B to a basis B' for V. * S.E.L

$$B' = \{\underline{b}_1, \dots, \underline{b}_{r+1}, \dots, \underline{b}_n\}$$

Set $\underline{c}_i := \varphi(\underline{b}_i) \quad 1 \leq i \leq r$

$\{\underline{c}_i\}$ are linearly independent because

if $\left(\sum_{i=1}^{r} k_i \underline{c}_i = \underline{0} \right.$ for $k_i \in \mathbb{F}$

then $\sum_{i=1}^{r} k_i \varphi(\underline{b}_i) = \underline{0}$

i.e. $\left. \varphi(k_1 \underline{b}_1 + \dots + k_r \underline{b}_r) = \underline{0} \right)$

then $k_1 \underline{b}_1 + \dots + k_r \underline{b}_r \in \ker(\varphi)$

& so $k_1 \underline{b}_1 + \dots + k_r \underline{b}_r = \sum_{i=r+1} j_i \underline{b}_i$

For some $j_i \in \mathbb{F}$

$\Rightarrow k_i, j_i = 0$ because $\{\underline{b}_i\}$ is a basis.

Ex: $\{\underline{c}_i\}$ forms a basis for $\mathrm{Im}(\varphi)$

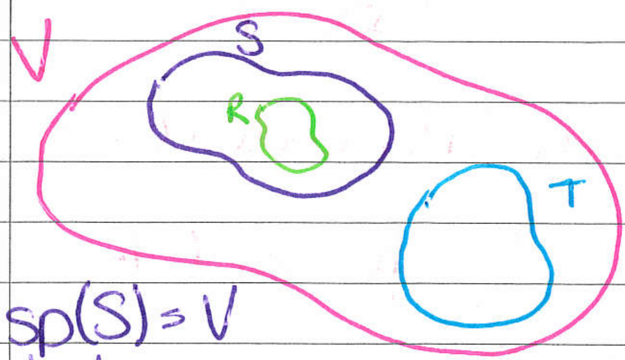
Extend $\{\underline{c}_i\}$ to a basis $\{\underline{c}_1, \ldots \underline{c}_r, \underline{c}_{r+1}, \ldots, \underline{c}_m\}$ for W.

Set $\underline{v}_i = \underline{b}_i$ for $i = 1, \ldots, n$

& $\underline{w}_i = \underline{c}_i$ for $i = 1, \ldots, m$

Ex: verify (i) - (iii)

Extend:



Conclude $\exists R \subseteq S$

① $m = |T| \leq |S| = n$

② $\exists R \subseteq S$ s.t. $sp(R \cup T) = V$

$sp(S) = V$

$|S| = n$

T lin indep.

$|T| = m$

$|R| = n - m$

$\Rightarrow |R \cup T| = n$

Ex: $V = \mathbb{R}^3$. $T = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\}$

$S = \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$ $R = \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$

check $sp(T \cup R)$ & conditions for S.E.L. Hold