Problems for MAS277 Vector Spaces and Fourier Theory

Chapter 3: Inner product spaces

1. Is the function $\langle f, g\rangle = f(0)g(0)$ an inner product on $\mathbb{R}[x]_{\leq 2}$?

2. If x and y are two vectors in an inner product space, show that
$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

3. Use the usual inner product $\langle A, B\rangle = \text{trace}(AB^T)$ on $M_3(\mathbb{R})$.

   (a) Calculate all the inner products $\langle C_i, C_j\rangle$, where
$$C_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}, \qquad C_3 = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}.$$

   (b) Show that if $A^T = A$ and $B^T = -B$ then A and B are orthogonal.

4. Use the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$ on $\mathbb{R}[x]_{\leq 2}$.

   (a) Find $\langle x+1, x^2+x\rangle$

   (b) Show that if $0 \leq i, j \leq 2$ and $i+j$ is odd then $\langle x^i, x^j\rangle = 0$.

   (c) Consider a polynomial $u(x) = px^2 + q$, and another polynomial $f(x) = ax^2 + bx + c$. Give a formula for $4f(-1) - 8f(0) + 4f(1)$ and another formula for $\langle f, u\rangle$. Hence find p and q such that $\langle f, u\rangle = 4f(-1) - 8f(0) + 4f(1)$ for all quadratic polynomials f.

5. In an inner product space, show that two vectors x and y are orthogonal if and only if
$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

6. Put
$$V = \{f \in C^\infty(\mathbb{R}) \mid f + f'' = 0\}.$$
   For $f, g \in V$ put
$$\langle f, g\rangle(t) = f(t)g(t) + f'(t)g'(t),$$
   so $\langle f, g\rangle \in C^\infty(\mathbb{R})$.

   (a) Prove that $\langle f, g\rangle$ is actually a constant. (Hint: Differentiate $\langle f, g\rangle$.)

   (b) Prove that if $f \in V$ then $f' \in V$, so that differentiation gives a linear map $D : V \longrightarrow V$.

   (c) The functions sin and cos give a basis for V. Using this, show that $\langle\ ,\ \rangle$ is an inner product on V.

   (d) What is the matrix representation of D with respect to the basis $\{\sin, \cos\}$?

7. Show that for any $f \in C[0,1]$ we have

$$\left|\int_0^1 (1+x)\, f(x)\, dx\right| \le \sqrt{\frac{7}{3}} \left(\int_0^1 f(x)^2\, dx\right)^{1/2}$$

Find a non-zero function $f \in C[0,1]$ for which the above inequality is actually an equality.

8. Show that for any $f \in C[-1,1]$ we have

$$\left|\int_{-1}^1 \sqrt{1-x^2}\, f(x)\, dx\right| \le \frac{2}{\sqrt{3}} \left(\int_{-1}^1 f(x)^2\, dx\right)^{1/2}$$

Find a non-zero function $f \in C[-1,1]$ for which the above inequality is actually an equality.

9. Show that for any $f \in C[0,1]$ we have

$$\left(\int_0^1 f(x)^3\, dx\right)^2 \le \left(\int_0^1 f(x)^2\, dx\right)\left(\int_0^1 f(x)^4\, dx\right).$$

For which functions f is this actually an equality?

10. Define an inner product on $\mathbb{R}[x]_{\le 2}$ by $\langle f, g\rangle = \int_0^1 f(x)g(x)\, dx$. Find a polynomial $f(x) = ax^2 + bx + c$ orthogonal to both of 1 and x.

11. Consider the Fourier space $C[-\pi, \pi]$ of continuous functions $[-\pi, \pi] \longrightarrow \mathbb{R}$ with the inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)\, dt.$$

Consider the subspace V of $C[-\pi,\pi]$ spanned by $\{1, \cos t, \sin t\}$, and define the linear map $\phi : V \longrightarrow V$ by $\phi(1) = 0$, $\phi(\cos t) = \sin t$, $\phi(\sin t) = \cos t$. Compute $\langle \phi(a + b\cos t + c\sin t), \alpha + \beta\cos t + \gamma \sin t\rangle$. If ϕ^* denotes the adjoint of ϕ, show that $\phi^* = \phi$.

12. Consider the map $\phi : \mathbb{R}^3 \longrightarrow M_{2\times 2}(\mathbb{R})$ given by $\phi\begin{pmatrix} x \\ y \\ z\end{pmatrix} = \begin{pmatrix} x & y \\ y & z\end{pmatrix}$. Let $\langle , \rangle$ be the standard inner product defined on $M_{2\times 2}(\mathbb{R})$ (see Question 3). Given a matrix $A = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$, find a vector $\mathbf{w} = (p, q, r)^T$ such that $\langle \phi(\mathbf{v}), A\rangle = \langle \mathbf{v}, \mathbf{w}\rangle$ for all vectors $\mathbf{v} \in \mathbb{R}^3$.

13. Consider the map $\phi : M_{3\times 3}(\mathbb{R}) \longrightarrow M_{3\times 3}(\mathbb{R})$ given by

$$\phi\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9\end{pmatrix} = \begin{pmatrix} 0 & a_4 & a_7 \\ 0 & 0 & a_8 \\ 0 & 0 & 0\end{pmatrix}.$$

Give a formula for the adjoint map $\phi^* : M_{3\times 3}(\mathbb{R}) \longrightarrow M_{3\times 3}(\mathbb{R})$.

14. Consider the map $\phi : M_{2\times 2}(\mathbb{R}) \longrightarrow M_{2\times 2}(\mathbb{R})$ given by $\phi(A) = QAQ$, where $Q = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Show that $\phi^* = \phi$.

15. In this exercise we give the space $\mathbb{R}[x]_{\leq 2}$ the inner product $\langle f, g\rangle = \int_{-1/2}^{1/2} f(x)g(x)\,dx$. Define $\chi : \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}$ by $\chi(f) = f''(0)$. If $f(x) = ax^2+bx+c$, what is $\chi(f)$? Find an element $u \in \mathbb{R}[x]_{\leq 2}$ such that $\chi(f) = \langle f, u\rangle$ for all f, and thus give a formula for χ^*.

16. Let V be an inner product space, and let $\phi : V \longrightarrow V$ be a linear map with the property that $\phi^*(\phi(\mathbf{v})) = \mathbf{v}$ for all $\mathbf{v} \in U$. Let $\mathcal{U} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthonormal set in V. Show that $\{\phi(\mathbf{v}_1), \ldots, \phi(\mathbf{v}_n)\}$ is an orthonormal set in V.

17. Define a map $\alpha : \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}[x]_{\leq 2}$ by $\alpha(f) = (3x^2 - 1)f''$. You may assume that if we use the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$ on $\mathbb{R}[x]_{\leq 2}$, then α is self-adjoint.

    (a) Show that $\alpha(\alpha(f)) = 6\alpha(f)$ for all f.

    (b) Deduce that if f is a non-zero eigenvector of α with eigenvalue λ, then $\alpha(\alpha(f)) = \lambda^2 f$ and $\lambda^2 = 6\lambda$.

    (c) Find an orthogonal basis for $\mathbb{R}[x]_{\leq 2}$ consisting of eigenvectors for α.

Chapter 4: Gram-Schmidt and Fourier Theory

1. Consider the Fourier space $C[-\pi, \pi]$ of continuous functions $[-\pi, \pi] \longrightarrow \mathbb{R}$ with the inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)\,dt.$$

   If k is an integer, what is $\langle t, \sin kt\rangle$?

2. Consider the Fourier space $C[-\pi, \pi]$ of continuous functions $[-\pi, \pi] \longrightarrow \mathbb{R}$ with the inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)\,dt.$$

   Define the angle between two vectors to between two vectors $\mathbf{u}, \mathbf{v}$ in an inner product space to be the unique $\theta \in [0, \pi]$ given by

$$\theta = \arccos\left(\frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}\right).$$

    (a) Compute the cosine of the angle between $\cos 3t$ and $\cos t \cos 4t$.

    (b) Show that $\cos 2t \sin t$ and $\cos 5t \sin t$ are orthogonal.

3. Consider the Fourier space $C[-\pi, \pi]$ of continuous functions $[-\pi, \pi] \longrightarrow \mathbb{R}$ with the inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)\,dt.$$

Show that for any $f \in C[-\pi, \pi]$, we have

$$\left| \int_{-\pi}^{\pi} \sin x \, f(x) \, dx \right| \leq \sqrt{\pi} \left(\int_{-\pi}^{\pi} f(x)^2 \, dx \right)^{1/2}$$

Find a non-zero function $f \in C[-\pi, \pi]$ for which the above inequality is actually an equality.

4. Find an orthogonal basis for the subspace of the inner product space $C[0, 1]$ (with its usual inner product) consisting of all polynomials of degree at most 2, using the Gram-Schmidt process on the basis 1, x, x^2.

5. Consider the Fourier space $C[-\pi, \pi]$ of continuous functions $[-\pi, \pi] \longrightarrow \mathbb{R}$ with the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)g(t) \, dt.$$

Consider the sequence $f_1(t) = \sin t$, $f_2(t) = \cos t \sin t$, $f_3(t) = \cos 2t \sin t$. Using the Gram-Schmidt method, find an orthogonal sequence g_1, g_2, g_3 such that $g_1 \in \mathrm{Sp}(f_1)$, $g_2 \in \mathrm{Sp}(f_1, f_2)$, $g_3 \in \mathrm{Sp}(f_1, f_2, f_3)$.

6. Put $V = \{B \in M_{2\times 2}(\mathbb{R}) \mid B^T = B\}$, and let $\pi : M_{2\times 2}(\mathbb{R}) \longrightarrow V$ be the orthogonal projection. Find an orthogonal basis for V, and use it to calculate $\pi(A)$ for an arbitrary matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Use this to show that $\pi(A) = (A + A^T)/2$.

7. Consider the space $V = M_4(\mathbb{R})$ with the usual inner product $\langle A, B \rangle = \mathrm{trace}(AB^T)$. Consider the following sequence in V:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Find an orthonormal sequence $C_1, \ldots, C_4$ in V such that $\mathrm{Sp}\{A_1, \ldots, A_i\} = \mathrm{Sp}\{C_1, \ldots, C_i\}$ for all i. (You can use the Gram-Schmidt procedure for this but it is easier to find an answer by inspection.)

8. Consider the following vectors in $\mathbb{R}^5$:

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_5 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Find an orthonormal sequence $\hat{v}_1, \ldots, \hat{v}_5$ such that $\mathrm{Sp}\{\hat{v}_1, \ldots, \hat{v}_i\} = \mathrm{Sp}\{u_1, \ldots, u_i\}$ for all i.