Problems for MAS277 Vector Spaces and Fourier Theory

The problems are organized according to Chapter. All the problems for tutorials and homeworks are here.

Chapter 1: Vector spaces

1. Go back and read the MAS201 notes. Remind yourselves of the definitions and main properties of vectors in $\mathbb{R}^n$: subspaces, linear independence, spanning sets and bases. We will treat all these topics again in MAS277, in a slightly more abstract setting, and with more complete proofs.

2. Which of the following subsets of $\mathbb{C}$, taken with the usual notion of addition and multiplication, are fields:

   (a) $\{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$

   (b) $\{a + bi : a, b \in \mathbb{Z}\}$

   (c) $\{a + b\sqrt[3]{2} : a, b \in \mathbb{Q}\}$ **Hint:** $2^{\frac{1}{3}}$ is irrational.

3. In a vector space V, show carefully that, with the usual rules, it follows that:

   (a) $(-1)0_V = 0_V$;

   (b) $\alpha.0_V = 0_V$ if $\alpha \in \mathbb{R}$;

   (c) $y + (x - y) = x$ for any two vectors $x, y \in V$.

4. Let V denote the set of all pairs (x, y) of real numbers. If $\mathbf{v} = (x, y)$ and $\mathbf{v}' = (x', y')$ are elements of V, define

$$\begin{aligned} \mathbf{v} + \mathbf{v}' &= (x + x', y + y') \\ \alpha.\mathbf{v} &= (\alpha x, 0) \qquad \text{(note the funny definition here!)} \\ 0 &= (0, 0) \\ -\mathbf{v} &= (-x, -y). \end{aligned}$$

   Is V a vector space with this funny definition of scalar multiplication? (You will need to check the "usual rules"; if all of them are verified, then V will be a vector space – but if any of them fail, then of course V will not be a vector space.)

5. Explain why none of the following is a vector space (with the obvious definition of addition and scalar multiplication).

   (a) $V_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) \;\middle|\; a \le b \le c \le d \right\}$;

   (b) $V_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x + y \text{ is an integer} \right\}$;

   (c) $V_3 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x^2 = y^2 \right\}$;

   (d) $V_4 = \{p \in \mathbb{R}[x] \,|\, p(0)p(1) = 0\}$.

6. Consider the set of all sequences $(a_0, a_1, a_2, \ldots)$ such that $a_{n+2} = \alpha a_{n+1} + \beta a_n$ for some fixed real numbers α and β. Show that the set of such sequences forms a vector space.

7. Consider the following subsets of $\mathbb{R}^3$:

$$\begin{aligned} U_1 &= \{(x,y,z)^T \mid x = 0\}; \\ U_2 &= \{(x,y,z)^T \mid \text{either } x = 0 \text{ or } y = 0\}; \\ U_3 &= \{(x,y,z)^T \mid x + y = 0\}; \\ U_4 &= \{(x,y,z)^T \mid x + y = 1\}. \end{aligned}$$

Which of them are subspaces?

8. Consider the following subsets of the vector space $\mathbb{R}[x]$ of all polynomials:

$$\begin{aligned} U_1 &= \{p \mid \deg(p) = 3\}; \\ U_2 &= \{p \mid p(0) = p(1)\}; \\ U_3 &= \{p \mid p(x) \geq 0 \text{ for } 0 \leq x \leq 1\}. \end{aligned}$$

Which of them are subspaces?

9. Consider the following subspaces of $\mathbb{R}^4$:

$$\begin{aligned} U &= \{(w,x,y,z)^T \mid w - x + y - z = 0\} \\ V &= \{(w,x,y,z)^T \mid w + x + y = 0 = x + y + z\} \\ W &= \{(u, u+v, u+2v, u+3v)^T \mid u, v \in \mathbb{R}\}. \end{aligned}$$

Find $U \cap V$, $U \cap W$ and $V \cap W$.

10. Consider the planes P, Q and R in $\mathbb{R}^3$ given by

$$\begin{aligned} P &= \{(x,y,z)^T \mid x + 2y + 3z = 0\}, \\ Q &= \{(x,y,z)^T \mid 3x + 2y + z = 0\}, \\ R &= \{(x,y,z)^T \mid x + y + z = 0\}. \end{aligned}$$

Find $P \cap Q \cap R$. This system of planes has an unusual feature, not shared by most other systems of three planes through the origin. What is it?

11. Which of the following subsets of $\mathbb{R}^4$ is a subspace?

$$\begin{aligned} U_1 &= \{(w,x,y,z)^T \mid w + x = 0\}; \\ U_2 &= \{(w,x,y,z)^T \mid w + x = 1\}; \\ U_3 &= \{(w,x,y,z)^T \mid w + 2x + 3y + 4z = 0\}; \\ U_4 &= \{(w,x,y,z)^T \mid w + x^2 + y^3 + z^4 = 0\}; \\ U_5 &= \{(w,x,y,z)^T \mid w^2 + x^2 = 0\}. \end{aligned}$$

12. Which of the following subsets of $F(\mathbb{R})$ are subspaces?

$$\begin{aligned} U_1 &= \{f \mid f(0) = 0\}, \\ U_2 &= \{f \mid f(1) = 1\}, \\ U_3 &= \{f \mid f(0) \geq 0\}, \\ U_4 &= \{f \mid f(0) = f(1)\}, \\ U_5 &= \{f \mid f(0)f(1) = f(2)f(3)\}. \end{aligned}$$

13. For each of the following vector spaces V, give an example of a subspace $W \leq V$ such that $W \neq 0$ and $W \neq V$.

   (a) $V = \mathbb{R}[x]_{\leq 3}$;

   (b) $V = M_{2\times 3}(\mathbb{R})$;

   (c) $V = \{(x,y,z)^T \in \mathbb{R}^3 \mid x+y+z=0\}$.

14. For each of the following vector spaces V, give an example of subspaces $U, W \leq V$ such that $U \neq 0$ and $W \neq 0$ but $U \cap W = 0$.

   (a) $V = \mathbb{R}^4$;

   (b) $V = M_{2\times 2}(\mathbb{R})$;

   (c) $V = \{(x,y,z)^T \in \mathbb{R}^3 \mid x+y+z=0\}$.

15. Inside $\mathbb{R}^2$, consider the subspaces $U = \{(x,0)^T \mid x \in \mathbb{R}\}$, $V = \{(0,y)^T \mid y \in \mathbb{R}\}$ and $W = \{(z,z)^T \mid z \in \mathbb{R}\}$. Show that

$$U \cap (V+W) \neq (U \cap V) + (U \cap W).$$

16. Put $V = \mathbb{R}[x]_{\leq 2}$ and $U = \{f \in V \mid f(0) = 0\}$ and $W = \{f \in V \mid f(1) + f(-1) = 0\}$. Show that $U \cap W$ is the set of all polynomials of the form $f(x) = bx$, and that $U + W = V$. Please write your argument carefully, using complete sentences and correct notation.

17. Put

$$L = \left\{ \begin{pmatrix} s \\ 2s \end{pmatrix} \;\middle|\; s \in \mathbb{R} \right\}, \qquad M = \left\{ \begin{pmatrix} 2t \\ t \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

   Show that $L \cap M = 0$ and $L + M = \mathbb{R}^2$ (or in other words, $\mathbb{R}^2 = L \oplus M$).

18. Put $V = M_3(\mathbb{R})$ and $U = \{A \in V \mid A^T = A\}$ and $W = \{A \in V \mid A^T = -A\}$. Show that $U \cap W = 0$ and $U + W = V$ (or in other words, $V = U \oplus W$).

19. For each of the following lists of vectors, say (with justification) whether they are linearly independent, whether they span $\mathbb{R}^3$, and whether they form a basis of $\mathbb{R}^3$. (If you understand the concepts involved, you should be able to do this by eye, without any calculation.)

   (a) $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix}, \mathbf{u}_4 = \begin{pmatrix} 7 \\ 0 \\ 8 \end{pmatrix}.$

   (b) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$

   (c) $\mathbf{w}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$

   (d) $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}, \mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}.$

20. Which of the following lists of vectors are linearly independent?

    (a) $\mathbf{u}_1 = (1,0,0,0,1)^T, \mathbf{u}_2 = (0,2,0,2,0)^T, \mathbf{u}_3 = (0,0,3,0,0)^T$
    (b) $\mathbf{v}_1 = (1,1,1,1)^T, \mathbf{v}_2 = (2,0,0,2)^T, \mathbf{v}_3 = (0,4,4,0)^T$
    (c) $\mathbf{w}_1 = (1,1,2)^T, \mathbf{w}_2 = (4,5,7)^T, \mathbf{w}_3 = (1,1,1)^T$

21. Consider $\mathbb{R}$ as a *rational* vector space. Prove that α and β are linearly independent if and only if α/β is irrational.

22. Find two bases of $\mathbb{R}^4$ such that the only vectors common to both are $(1,1,0,0)^T$ and $(0,0,1,1)^T$.

23. Which of the following lists of matrices spans $M_{2\times 2}(\mathbb{R})$?

    (a) $\mathcal{A} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$
    (b) $\mathcal{B} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
    (c) $\mathcal{C} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
    (d) $\mathcal{D} = \begin{pmatrix} 463 & 859 \\ 265 & -463 \end{pmatrix}, \begin{pmatrix} 937 & 724 \\ 195 & -937 \end{pmatrix}, \begin{pmatrix} 431 & 736 \\ 428 & -431 \end{pmatrix}, \begin{pmatrix} 777 & 152 \\ 522 & -777 \end{pmatrix}$

24. Put $s_k(x) = (x+k)^2$. Prove that the list s_0, s_1, s_2 spans $\mathbb{R}[x]_{\leq 2}$.

25. Suppose we have real numbers $a, b, c \in \mathbb{R}$ and functions $f, g, h \in C(\mathbb{R})$ such that

$$\begin{aligned} f(a) &= 1, & g(a) &= 0, & h(a) &= 0; \\ f(b) &= 0, & g(b) &= 1, & h(b) &= 0; \\ f(c) &= 0, & g(c) &= 0, & h(c) &= 1. \end{aligned}$$

    Prove that f, g and h are linearly independent.

26. Consider the vector space $\mathcal{F}(\mathbb{R}, \mathbb{R})$ of functions $\mathbb{R} \longrightarrow \mathbb{R}$.

    (a) If α is some fixed real number, show that the three functions $\sin x$, $\cos x$ and $\sin(x+\alpha)$ are linearly dependent.
    (b) If $k > 1$ is some fixed integer, show that the three functions $\sin x$, $\cos x$ and $\sin kx$ are linearly independent.

27. Define subspaces V, W of $\mathbb{R}[x]_{\leq 3}$ by

$$\begin{aligned} V &= \{f \in \mathbb{R}[x]_{\leq 3} \mid f(x) + f(-x) = 0\}, \\ W &= \{f \in \mathbb{R}[x]_{\leq 3} \mid f''(1) = 2f'(1) = 6f(1)\}. \end{aligned}$$

    Find bases for V, W and $V \cap W$. Prove that

$$V + W = \{f \in \mathbb{R}[x]_{\leq 3} \mid f''(0) = 6f(0)\}.$$

28. Let V be a finite-dimensional vector space, and let U and W be subspaces of V. In lectures we proved that there exist elements

$$u_1, \dots, u_p, v_1, \dots, v_q, w_1, \dots, w_r$$

such that

- $u_1, \dots, u_p$ is a basis for $U \cap W$
- $u_1, \dots, u_p, v_1, \dots, v_q$ is a basis for U
- $u_1, \dots, u_p, w_1, \dots, w_r$ is a basis for W
- $u_1, \dots, u_p, v_1, \dots, v_q, w_1, \dots, w_r$ is a basis for $U + W$.

Find elements as above for the case $V = M_2(\mathbb{R})$ and $U = \{A \in V \mid A^T = A\}$ and $W = \{A \in V \mid \operatorname{trace}(A) = 0\}$.

29. Let Z be a finite-dimensional vector space, and let U, V and W be subspaces of Z. Suppose that

$$\begin{aligned} \dim(U) &= 2, \quad \dim(U \cap V) = 1, \\ \dim(V) &= 3, \quad \dim(V \cap W) = 2, \\ \dim(W) &= 4, \quad \dim((U + V) \cap W) = 3. \end{aligned}$$

Find the dimensions of $U+V$, $V+W$ and $U+V+W$. Hence show that $U+V+W = V+W$ and thus that U is a subspace of $V+W$.

30. What is the dimension of $\mathbb{C}$, viewed as a vector space over $\mathbb{R}$? And what would your answer be if we viewed it as a vector space over $\mathbb{C}$?

31. (a) Call a 3×3 matrix a *semi-magic* square if all its rows and columns (but not necessarily its diagonals) have the same sum. Let V be the set of semi-magic squares. Without writing out the laborious details, persuade yourself that V is a subspace of $M_3(\mathbb{R})$; find a basis, and hence give its dimension.
    (b) You should find that the dimension of the space of semi-magic squares is greater than that of the space of magic squares. Find a semi-magic square which is not magic.
    (c) Let's say that a semi-magic square is *nearly magic* if both diagonals have the same sum (which may or may not be the same as the row and column sum). Again persuade yourself that the set U of nearly magic squares is a subspace of V. What is its dimension? Are there nearly magic squares which are not magic? If so, find one.

32. Consider the vector space $\mathbb{R}^2$. Write

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Then $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^2$. Suppose that the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ can be written as $\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2$. Give a matrix $A \in M_2(\mathbb{R})$ such that

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = A \begin{pmatrix} a \\ b \end{pmatrix}.$$

Verify this explicitly.

33. Consider the vector space $\mathbb{R}^2$. Write

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_4 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Then $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{V}' = \{\mathbf{v}_3, \mathbf{v}_4\}$ are both bases of $\mathbb{R}^2$. Suppose that the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ can be written as $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2$ and $\alpha_1'\mathbf{v}_3 + \alpha_2'\mathbf{v}_4$. Give a matrix $A \in M_2(\mathbb{R})$ such that

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = A \begin{pmatrix} \alpha_1' \\ \alpha_2' \end{pmatrix}.$$

Verify this explicitly.

34. Consider the vector space $\mathbb{R}^2$. Write

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{V}' = \{\mathbf{v}_2, \mathbf{v}_3\}$ are both bases of $\mathbb{R}^2$. Suppose that the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ can be written as $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2$ and $\alpha_1'\mathbf{v}_2 + \alpha_2'\mathbf{v}_3$ (i.e., as $\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}_{\mathcal{V}}$ and $\begin{bmatrix} \alpha_1' \\ \alpha_2' \end{bmatrix}_{\mathcal{V}'}$. Give a matrix $A \in M_2(\mathbb{R})$ such that

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = A \begin{pmatrix} \alpha_1' \\ \alpha_2' \end{pmatrix}.$$

Verify this explicitly.

Chapter 2: Linear maps

1. Consider the following maps $\mathbb{R}^2 \longrightarrow \mathbb{R}$. Which are linear maps?

   (a) $\phi_1((x,y)^T) = x$;
   (b) $\phi_2((x,y)^T) = y+1$;
   (c) $\phi_3((x,y)^T) = x^2$;
   (d) $\phi_4((x,y)^T) = x+2y$;
   (e) $\phi_5((x,y)^T) = \sqrt{x^2+y^2}$.

2. Consider the following maps $\mathbb{R}^3 \longrightarrow \mathbb{R}$. Which are linear maps?

   (a) $\phi_1((x,y,z)^T) = x+y$;
   (b) $\phi_2((x,y,z)^T) = x-z^2$;
   (c) $\phi_3((x,y,z)^T) = z-1$;
   (d) $\phi_4((x,y,z)^T) = xyz$;
   (e) $\phi_5((x,y,z)^T) = x-2y+3z$.

3. Consider the following maps $\mathbb{R}[x] \longrightarrow \mathbb{R}$. Which are linear maps?

   (a) $\phi_1(f) = \int_{-1}^{2} f(x)\,dx$;
   (b) $\phi_2(f) = \int_{0}^{2} f^2(x)\,dx$;
   (c) $\phi_3(f) = \int_{0}^{1} xf(x)\,dx$;
   (d) $\phi_4(f) = \int_{0}^{1} f(x^2)\,dx$;
   (e) $\phi_5(f) = f(0) + f'(1) + f''(2)$;
   (f) $\phi_6(f) = f(0)f(1)$.

4. Is the map $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $\phi((x,y)^T) = (x+y, x-y)^T$ linear?

5. Show that the map $\mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ given by $\phi(f)(x) = f(x+1)$ is linear.

6. In each of the cases below, give an example of a non-zero linear map $\phi : V \longrightarrow W$. (Here, "non-zero" means that there is at least one $v \in V$ such that $\phi(v) \neq 0$.)

   (a) $V = \mathbb{R}^4$, $W = \mathbb{R}^2$;
   (b) $V = M_3(\mathbb{R})$, $W = \mathbb{R}^2$;
   (c) $V = M_3(\mathbb{R})$, $W = \mathbb{R}[x]$;
   (d) $V = \mathbb{R}[x]$, $W = M_2(\mathbb{R})$.

7. Let V be a vector space over $\mathbb{R}$, and let $\phi : \mathbb{R}[x]_{\leq 1} \longrightarrow V$ be a linear map. Show that there exist elements $u, v \in V$ such that

   $$\phi(ax+b) = au + bv$$

   for all $a, b \in \mathbb{R}$.

8. Let V and W be vector spaces, and let $\phi : V \longrightarrow W$ be a linear map. Let $\mathcal{V} = v_1, \ldots, v_n$ be a list of elements of $\mathcal{V}$.

(a) Show that if $v_1, \ldots, v_n$ are linearly dependent, then so are $\phi(v_1), \ldots, \phi(v_n)$.

(b) Give an example where $v_1, \ldots, v_n$ are linearly independent, but $\phi(v_1), \ldots, \phi(v_n)$ are linearly dependent.

(c) Show that if $\phi(v_1), \ldots, \phi(v_n)$ are linearly independent, then $v_1, \ldots, v_n$ are linearly independent.

9. Given vectors $(p, q)^T, (r, s)^T \in \mathbb{R}^2$, we can define a linear map $\phi : M_2(\mathbb{R}) \longrightarrow \mathbb{R}$ by

$$\phi(A) = \begin{pmatrix} p & q \end{pmatrix} A \begin{pmatrix} r \\ s \end{pmatrix}.$$

Show that p, q, r and s *cannot* be chosen so that $\phi(A) = \mathrm{trace}(A)$ for all $A \in M_{2\times 2}(\mathbb{R})$.

10. Define $\phi : \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}[x]_{\leq 2}$ by $\phi(f) = f + f' + f''$. Find the matrix representation of ϕ with respect to the basis $\{1, x, x^2\}$. What is the trace and determinant of this matrix?

11. Define $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by

$$\phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y + z \\ z + x \\ x + y \end{pmatrix}.$$

Find the matrix representation of ϕ with respect to the standard basis of $\mathbb{R}^3$. Then find the matrix representation of ϕ with respect to the basis

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{u}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \qquad \mathbf{u}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

12. Define a linear map $\phi : \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}^3$ by $\phi(f) = (f(0), f'(1), f''(2))^T$. What is the matrix representation of ϕ with respect tothe usual bases of $\mathbb{R}[x]_{\leq 2}$ and $\mathbb{R}^3$?

13. Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$. Let $\kappa : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be defined by $\kappa(\mathbf{v}) = \mathbf{a}.\mathbf{v}$, and $\lambda : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $\lambda(\mathbf{v}) = \mathbf{a} \times \mathbf{v}$. Find matrices K and L such that $\kappa = \phi_K$ and $\lambda = \phi_L$.

14. Show that the space $V_{\leq k}$ of all polynomials of degree at most k and with constant term equal to 0 is a subspace of $\mathbb{R}[x]$. Define the map $i(x^k) = \frac{1}{k+1}x^{k+1}$ (essentially this is integration – but we choose to omit any constant of integration), and regard it as a map $i : \mathbb{R}[x]_{\leq 4} \longrightarrow V_{\leq 5}$, with bases $\{1, x, x^2, x^3, x^4\}$ and $\{x, x^2, x^3, x^4, x^5\}$ respectively. What is the matrix representation of i?

15. For any continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$, we define $D(f) = f'$. This restricts to a map $D : \mathbb{R}[x]_{\leq 3} \longrightarrow \mathbb{R}[x]_{\leq 2}$. If we consider these as vector spaces with bases $\{1, x, x^2, x^3\}$ and $\{1, x, x^2\}$ respectively, give the matrix representation for D. Guess at the answer if we had instead restricted to $D : \mathbb{R}[x]_{\leq 4} \longrightarrow \mathbb{R}[x]_{\leq 3}$, with the obvious bases.

16. For any continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$, we write

$$I(f) = \int_0^1 f(x)\, dx \in \mathbb{R}.$$

This defines a map $I : C(\mathbb{R}) \longrightarrow \mathbb{R}$. If we consider this just as a map on $\mathbb{R}[x]_{\leq 4}$, with basis $\{1, x, x^2, x^3, x^4\}$, give the matrix representation for I.

17. Define maps $\alpha, \beta : M_{2\times 2}(\mathbb{R}) \longrightarrow M_{2\times 2}(\mathbb{R})$ by

$$\alpha(X) = X - X^T, \qquad \beta(X) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} X.$$

Put $\mathcal{E} = E_1, E_2, E_3, E_4$, where

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let A be the matrix of α with respect to the basis $\mathcal{E}$, let B be the matrix of β with respect to $\mathcal{E}$, and let C be the matrix of $\alpha\beta$ with respect to $\mathcal{E}$.

(a) Find $\alpha(E_i)$ for each i, and hence find A.

(b) Find $\beta(E_i)$ for each i, and hence find B.

(c) Find $\alpha(\beta(E_i))$ for each i, and hence find C.

(d) Check that $C = AB$.

18. If the matrix representation of a linear map $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with respect to the standard basis $\{(1,0)^T, (0,1)^T\}$ is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, what is the matrix representation of ϕ with respect to the basis $\{(1,1)^T, (1,-1)^T\}$?

19. If the matrix representation of a linear map $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with respect to the standard basis $\{(1,0,0)^T, (0,1,0)^T, (0,0,1)^T\}$ is $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$, what is the matrix representation of ϕ with respect to the basis $\{(0,1,-1)^T, (1,-1,1)^T, (-1,1,0)^T\}$?

20. Fix a real number λ, and let V be the set of functions of the form

$$f(x) = (ax^2 + bx + c)e^{\lambda x}.$$

In other words, we have $V = \mathbb{R}[x]_{\leq 2} e^{\lambda x}$.

(a) Write down a basis for V.

(b) Show that if $f \in V$ then $f' \in V$, so we can define a linear map $D : V \longrightarrow V$ by $D(f) = f'$.

(c) What is the matrix representation of D with respect to your chosen basis?

(d) Show that $(D - \lambda)^3(f) = 0$ for all $f \in V$.

21. For each of the following linear maps, decide whether the map is injective, whether it is surjective, and whether it is an isomorphism. Please write your arguments carefully, using complete sentences and correct notation. Where counterexamples are required, make them as simple and specific as possible.

   (a) $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by $\phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ x \end{pmatrix}$;

   (b) $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by $\phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y \\ y-z \end{pmatrix}$;

   (c) $\phi : \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}^3$ given by $\phi(f) = (f(0), f'(0), f''(0))^T$;

   (d) $\phi : \mathbb{R}^2 \longrightarrow M_2(\mathbb{R})$ given by $\phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & x+y \\ x+y & 0 \end{pmatrix}$;

   (e) $\phi : \mathbb{R}[x] \longrightarrow \mathbb{R}$ given by $\phi(f) = \int_{-1}^{1} f(x)\, dx$

22. Let V be the set of all sequences $(a_0, a_1, a_2, \ldots)$ of real numbers for which $a_{n+2} = 3a_{n+1} - 2a_n$ for all n.

   (a) Define $\pi : V \longrightarrow \mathbb{R}^2$ by

   $$\pi(a_0, a_1, a_2, \ldots) = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}.$$

   Show that $\ker(\pi) = 0$, so that π is injective.

   (b) Define sequences $u = (u_n)$ and $v = (v_n)$ by $u_n = 1$, $v_n = 2^n$, for all n. Show that u and v are in V.

   (c) Find constants $\alpha, \beta, \gamma, \delta$ such that the sequences $b = \alpha u + \beta v$ and $c = \gamma u + \delta v$ satisfy $\pi(b) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\pi(c) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

   (d) Show that b and c give a basis for V, and deduce that u and v give a basis for V.

   (e) Define $\phi : V \longrightarrow V$ by

   $$\phi(a_0, a_1, a_2, \ldots) = (a_1, a_2, a_3, \ldots).$$

   What is the matrix of ϕ with respect to the basis $\{u, v\}$?

23. Define $\phi : \mathbb{R}^2 \longrightarrow M_{2\times 2}(\mathbb{R})$ by $\phi(\begin{pmatrix} u \\ v \end{pmatrix}) = \begin{pmatrix} u & -u \\ -v & v \end{pmatrix}$. Show that ϕ is injective, and that

$$\operatorname{im}(\phi) = \left\{ A \in M_2(\mathbb{R}) \;\middle|\; A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

24. Define $\phi : M_{2\times 2}(\mathbb{R}) \longrightarrow M_{2\times 2}(\mathbb{R})$ by $\phi(A) = A - \frac{1}{2}\operatorname{trace}(A)I$. Show that $\ker(\phi) = \{aI \mid a \in \mathbb{R}\}$ and that $\operatorname{im}(\phi) = \{A \in M_{2\times 2}(\mathbb{R}) \mid \operatorname{trace}(A) = 0\}$.

25. Define $\phi : \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}^3$ by

$$\phi(f) = \left(\int_{-1}^{0} f(x)\, dx, \int_{-1}^{1} f(x)\, dx, \int_{0}^{1} f(x)\, dx \right)^T.$$

(a) If $f(x) = ax^2 + bx + c$, find $\phi(f)$.
(b) Show that $\ker(\phi) = \{c(1 - 3x^2) \mid c \in \mathbb{R}\}$.
(c) Find a function $g_+(x) = px + q$ such that $\phi(g_+) = (1, 1, 0)^T$.
(d) Put $g_-(x) = g_+(-x)$, and show that $\phi(g_-) = (0, 1, 1)^T$.
(e) Deduce that $\text{im}(\phi) = \{(u, v, w)^T \in \mathbb{R}^3 \mid v = u + w\}$.

26. (a) Define a map $\phi : \mathbb{R}[x]_{\leq 3} \longrightarrow \mathbb{R}^2$ by $\phi(f) = (f(0), f(1))^T$. Show that this is surjective, and that the kernel is spanned by $x^2 - x$ and $x^3 - x^2$.
(b) Define a map $\psi : \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}^4$ by $\psi(f) = (f(0), f(1), f(2), f(3))^T$. Show that this is injective, and that the image is the space

$$V = \{(u_0, u_1, u_2, u_3)^T \in \mathbb{R}^4 \mid u_0 - 3u_1 + 3u_2 - u_3 = 0\}.$$

Find bases of $\mathbb{R}[x]_{\leq 2}$ and $\mathbb{R}^4$ with respect to which ψ has matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

27. Put $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M_2(\mathbb{R})$. Define $\phi : \mathbb{R}[x]_{\leq 2} \longrightarrow M_2(\mathbb{R})$ by $\phi(f) = f(J)$, or in other words

$$\phi(ax^2 + bx + c) = aJ^2 + bJ + cI.$$

Find bases for $\ker(\phi)$ and $\text{im}(\phi)$.

28. Choose real numbers a, b, c, d, and define the linear map $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $\phi(x, y) = (ax + by, cx + dy)$. Show that ϕ is an isomorphism if and only if $ad - bc \neq 0$.

29. Define a linear map $\phi : M_{3\times3}(\mathbb{R}) \longrightarrow \mathbb{R}[x]_{\leq 4}$ by

$$\phi(A) = (1 \;\; x \;\; x^2) A \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}.$$

Show that ϕ is surjective, and find a basis for the kernel.

30. If $\phi : V \longrightarrow W$ is an isomorphism, show that the inverse map $\phi^{-1} : W \longrightarrow V$ is also linear.

31. Find the kernel and image of the differentiation map $\mathbb{R}[x]_{\leq n} \longrightarrow \mathbb{R}[x]_{\leq n}$.

32. Give an example of a linear map $\mathbb{R}[x]_{\leq 3} \longrightarrow \mathbb{R}[x]_{\leq 3}$ with a 2-dimensional kernel.

33. What is the rank and nullity of the linear map $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by the following matrices?

$$\text{(a)} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}; \quad \text{(b)} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \text{(c)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\text{(d)} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \text{(e)} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

34. Let $\phi : V \longrightarrow V$ be a linear map from a finite dimensional vector space V to itself. Show that the following are equivalent:

   (a) ϕ is injective;

   (b) ϕ is surjective;

   (c) ϕ is an isomorphism;

   (d) $\det A \neq 0$ if ϕ is represented by a matrix A with respect to some basis.

35. Let V and W be two vector spaces. Show that the collection $\mathcal{L}(V, W)$ of linear maps from V to W itself has the structure of a vector space. If $\dim(V) = m$ and $\dim(W) = n$, what is $\dim \mathcal{L}(V, W)$?