10/3

Recall: Let $L: V \to W$ be a map between vector spaces over a field $\mathbb{F}$. The following are true:

a) L is linear iff $L(\underline{u}+c\underline{v}) = L(\underline{u}) + cL(\underline{v})$ $\forall$ $\underline{u}, \underline{v} \in V$ & $c \in \mathbb{F}$

b) If $B = \{\underline{v}_i\}$ is a basis for V & $\{\underline{w}_i\} \subseteq W$ then there is a unique linear map T with $T(\underline{v}_i) = (\underline{w}_i)$

Ex $L: V \to W$ defined by $\underline{v} \mapsto \underline{0}_W$

$I: V \to V$ defined by $\underline{v} \mapsto \underline{v}$ identity

Obs if $\{\underline{b}_1, \ldots, \underline{b}_n\}$ is a basis for V then I_V is the unique linear map with $\underline{b}_i \mapsto \underline{b}_i$

HOMEWORK Ch1 Q 27, 29 Ch2, Q3
Due 11/03

Definition A linear transformation $L: V \to W$ between vector spaces is said to be invertible if there exists a map $L': W \to V$ s.t. $L \circ L' = I_W$ & $L' \circ L = I_V$

Prop Let $L: V \to W$ be a linear map. If L is invertible then the inverse map L' is a linear transformation & is unique.

__Proof__ $L: V \to W$ is linear & invertible
ie $\exists\ L': W \to V$ s.t. $L \cdot L' = I_W$
& $L' \circ L = I_V$

In particular L & L' are maps between sets. From MAS114, L is unique & L is a bijection.

To show L' is linear we let $\underline{w}, \underline{z} \in W$ & $c \in \mathbb{F}$

We know $\underline{w} = L(\underline{u})$ & $\underline{z} = L(\underline{v})$ for some $\underline{u}, \underline{v} \in V$ because L is a bijection.

$$L'(\underline{w} + c\underline{z}) = L'\big(L(\underline{u}) + cL(\underline{v})\big)$$

L is linear

$$\xrightarrow{\text{L is linear}} = L'\big(L(\underline{u} + c\underline{v})\big) = L' \circ L\,(\underline{u} + c\underline{v})$$

linearity

$$= I_V(\underline{u} + c\underline{v}) \overset{\text{linearity}}{=} \underline{u} + c\underline{v}$$

$$= L'(\underline{w}) + cL'(\underline{z})$$

because $L(\underline{u}) = \underline{w} \Rightarrow L' \circ L(\underline{u}) = L'(\underline{w})$

& similarly $L(\underline{v}) = \underline{z} \Rightarrow L' \circ L(\underline{v}) = L'(\underline{z})$

__Notation__ We call L' the inverse of L & write it as L^{-1}.

Def Two vector spaces V & W are said to be isomorphic if there exists an invertible linear map $L: V \to W$. When V & W are isomorphic we denote this by writing $V \cong W$.

Remark $L: V \to W$ groups

$$L(\underline{u} * \underline{v}) = L(\underline{u}) * L(\underline{v})$$

Ex $\mathbb{R}[x]_{\geq n} \cong \mathbb{R}^{n+1}$

Proof Consider $L: \mathbb{R}[x]_{\geq n} \to \mathbb{R}^{n+1}$

$$c_0 + c_1 x + \ldots + c_n x^n \mapsto (c_0 \ c_1 \ \ldots \ c_n)^T$$

(**Exercise**: Show this is linear)

Using ☆

~~Let L be the map~~

& so is $(c_0 \ c_1 \ \ldots \ c_n)^T \overset{L'}{\mapsto} c_0 + c_1 x + \ldots + c_n x^n$

$$L \circ L' = I_{\mathbb{R}^{n+1}} \quad \& \quad L' \circ L = I_{\mathbb{R}[x]_{\geq n}}$$

Hence, $\mathbb{R}[x]_{\geq n} \cong \mathbb{R}^{n+1}$

Theorem If V is a vector space of dimension n over a field $\mathbb{F}$ then $V \cong \mathbb{F}^n$.

Proof Let $\{\underline{v}_1, \dots, \underline{v}_n\}$ be a basis for V. & $\{\underline{e}_1, \dots, \underline{e}_n\}$ to be the standard basis for $\mathbb{F}^n$.

Define L to be the unique linear map taking $\underline{v}_i \mapsto \underline{e}_i$

& T to be the unique linear map taking $\underline{e}_i \mapsto \underline{v}_i$

Then $T \circ L = I_V$ but I_V is the unique linear map sending $\underline{v}_i \mapsto \underline{v}_i$

$L \circ T = I_{\mathbb{F}^n}$. Hence $T = L^{-1}$ & $V \cong \mathbb{F}^n$

Exercise $\mathbb{C}$ over $\mathbb{R}$ is isomorphic to $\mathbb{R}^2$ as vector spaces.

Ex Consider $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ & $\phi_A : \mathbb{R}^2 \to \mathbb{R}^2$
$\phi : \begin{pmatrix} x \\ y \end{pmatrix} \to A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 0 \end{pmatrix}$

Def Let $L : V \to W$ be a linear transformation between finite dimensional vector spaces. We define the kernel of L, denoted $\ker(L)$, & the image of L, denoted $\operatorname{im}(L)$, as follows:

* $\ker(L) := \{\underline{v} \in V \mid L(\underline{v}) = \underline{0}\}$
* $\operatorname{im}(L) := \{\underline{w} \in W \mid \underline{w} = L(\underline{v}) \text{ for some } \underline{v} \in V\}$

__Theorem__ If $L: V \to W$ is a linear transformation between vector spaces then $\ker(L) \subset V$ & im$(L) \subset W$ are subspaces.

__Proof__ Let $L: V \to W$ be linear & let $\underline{u}, \underline{v} \in \ker(L)$ (ie. $L(\underline{u}) = L(\underline{v}) = \underline{0}_w$) & $c \in \mathbb{F}$.

Need to show
- $\underline{0}_v \in \ker(L)$
- $\underline{u} + c\underline{v} \in \ker(L)$

We know $L(\underline{0}_v) = \underline{0}_w$ (already done this)

$L(\underline{u} + c\underline{v}) = L(\underline{u}) + cL(\underline{v})$ (by linearity)

$$= \underline{0}_w + c\,\underline{0}_w = \underline{0}_w$$

ie. $\underline{u} + c\underline{v} \in \ker(L)$ & hence $\ker(L)$ is a subspace.

__Note__ (im(L)) $\underline{0}_w = L(\underline{0}_v)$ (from before)

$\Rightarrow \underline{0}_w \in$ im(L)

if $\underline{x}, \underline{y} \in$ im(L) & $c \in \mathbb{F}$ then

$\underline{x} = L(\underline{u})$ & $\underline{y} = L(\underline{v})$ for some $\underline{u}, \underline{v} \in V$

ie. $\underline{x} + c\underline{y} = L(\underline{u}) + cL(\underline{v}) \overset{\text{by linearity}}{=} L(\underline{u} + c\underline{v})$

but $\underline{w} = \underline{u} + c\underline{v} \in V$. ie. $\underline{x} + c\underline{y} = L(\underline{w}) \in$ im(L)

& hence im(L) is a subspace of W.