Recall: $\dim(V) = |B|$ when B is a basis for V.

What about $V = \{\underline{0}\}$

✱ Define $\dim(\{\underline{0}\}) = 0$ ✱

Alternatively, we can change the definition of the span(U)

Ex: $sp(U) = \cap V_i$ where V_i is a subspace containing U

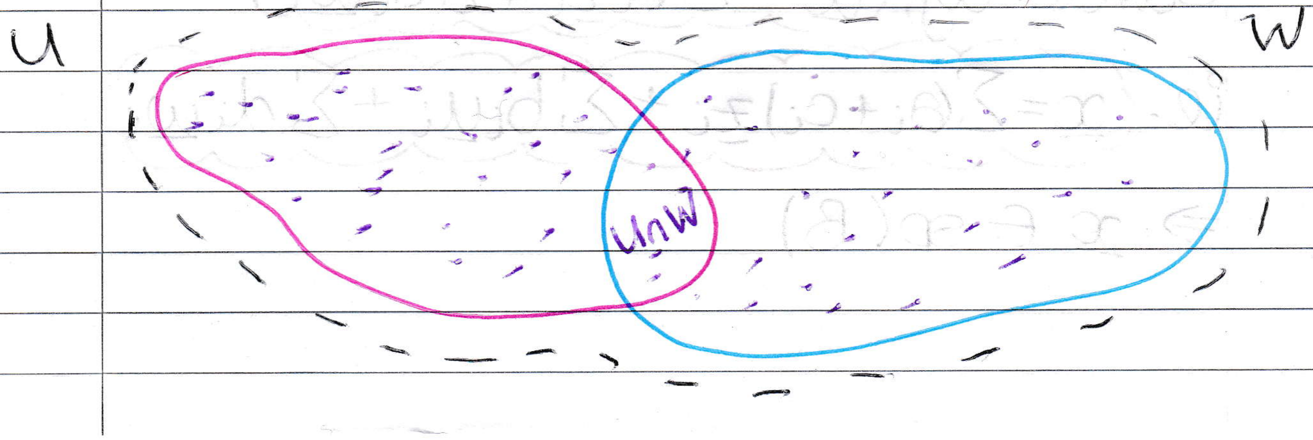
Do the top one.

Proposition Let V be an n-dimensional vector space. If S is a linearly independent subset of V with k elements then there exists a basis B for V such that $S \subseteq B$.

Proposition Let V be a vector space & let U, W be finite dimensional subspaces of V. The following are true.

a) $\dim(U) \leq \dim(W)$ with equality iff $U = V$

b) $\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$ ☆



$U+W := \{\underline{x} \mid \underline{x} = \underline{u} + \underline{w}$ for some $\underline{u} \in U$ & $\underline{w} \in W\}$

ie. $\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$

We let $B_1 = \{\underline{z}_1, \dots, \underline{z}_k\}$ be a basis ~~for~~ for $U \cap W$.

$U \cap W \subseteq U$ & so we extend B_1 to a basis for U.

ie. set $B_2 = \{\underline{z}_1, \dots, \underline{z}_k, \underline{u}_1, \dots, \underline{u}_n\}$ to be a basis for U.

& similarly set $B_3 = \{\underline{z}_1, \dots, \underline{z}_k, \underline{w}_1, \dots, \underline{w}_m\}$ to be a basis for W.

We will show that
$B = \{\underline{z}_1, \dots, \underline{z}_k, \underline{u}_1, \dots, \underline{u}_n, \underline{w}_1, \dots, \underline{w}_m\}$
is a basis for $U+W$.

<u>Note</u> that $\underline{x} \in U+W$ then
$\underline{x} = \underline{u} + \underline{w}$ for some $\underline{u} \in U$ & $\underline{w} \in W$.
So, $\underline{u} = \sum a_i \underline{z}_i + \sum b_i \underline{u}_i$ for some $a_i, b_i \in \mathbb{F}$ because B_2 is a basis for U.

Similarly, $\underline{w} = \sum c_i \underline{z}_i + \sum d_i \underline{w}_i$

ie. $\underline{x} = \sum (a_i + c_i) \underline{z}_i + \sum b_i \underline{u}_i + \sum d_i \underline{w}_i$

$\Rightarrow \underline{x} \in sp(B)$

Suppose there exists $\alpha_i, \beta_i, \gamma_i \in \mathbb{F}$ such that $\left(\sum \alpha_i \underline{z}_i + \sum \beta_i \underline{u}_i + \sum \gamma_i \underline{w}_i = \underline{0}\right)$ ✱

$\Rightarrow \left(\underbrace{\sum \gamma_i \underline{w}_i}_{\in W}\right) = \left(-\sum \alpha_i z_i - \sum \beta_i u_i \in U\right)$

$\Rightarrow \underline{w} \in U \cap W$

ie. $\underline{w} = \sum \gamma_i \underline{w}_i = \sum \delta_i \underline{z}_i$ because B_1 is a basis for $U \cap W$

$\Rightarrow \gamma_i \delta_i = 0$ because B_3 is a basis (ie. linearly independent)

Using $\gamma \delta_i = 0$ in ✱ implies $\alpha_i, \beta_i = \underline{0}$ because B_2 is a basis.

Hence, B is linearly independent so it is a basis.

☆ holds because $k+n+m = (k+n)+(k+m)-k$

Ex Let $\dim(U)=3$, $\dim(W)=4$, $\dim(U \cap W)=1$. Find $\dim(U+W)$

$\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$

$= 3+4-1 = \underline{\underline{6}}$

Ex If $U, W \subseteq V$ are subspaces with $\dim(U)=3$, $\dim(W)=5$. Find all possible dimensions of $U+W$.

Review

$C[a,b]$ $\mathbb{R}[x]_{\leq n}$ $M_{n \times m}(\mathbb{F})$

- Fields & Vector Spaces $\mathbb{R}^n$ $\mathbb{Q}$ $\mathbb{C}$ $\mathbb{R}$
- Subspaces, $U+W$, $U \oplus W$, $U \cap W$.
- $\underline{0} \in U$
  - $\underline{u} + c\underline{v} \in U$ $\quad \underline{u}, \underline{v} \in U$, $c \in \mathbb{F}$

U is a vector space.
If we want to show $U = \{f \in C[0,1] \mid \text{condition}\}$ is a vector space, show it is a subspace.
We can show U is a subspace of $C[0,1]$ & conclude U is a vector space
$U \oplus W = U + W$ when $U \cap W = \{\underline{0}\}$
$\dim(U \oplus W) = \dim(U) + \dim(W)$

- linear (in)dependence
- basis & dimension.