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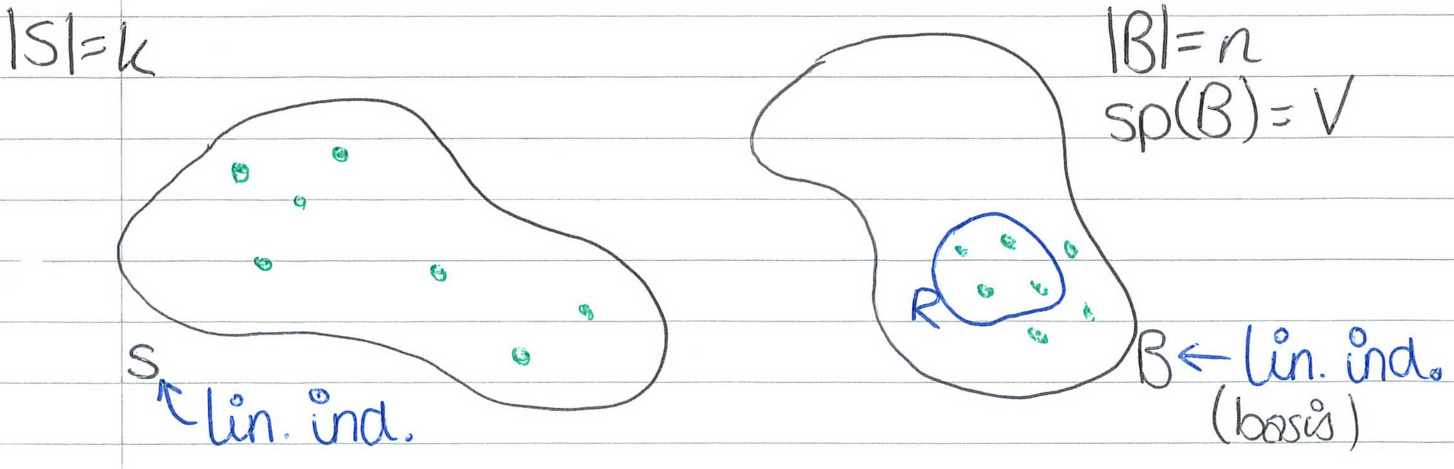
~~Ex~~ Remark One way to find the dimension of a vector space is to exhibit an explicit basis.

Ex

Vector Space	B = Basis lin. ind. spans	Dimension
$\{\underline{0}\}$	?	0
$\mathbb{R}^n$	$\{e_1, \dots, e_n\}$ $= (0 \dots 0, 1, 0, \dots, 0)$ (nth entry)	n
$\mathbb{R}[x]_n$	$\{1, x, \dots, x^n\}$	
$M_{n \times m}(\mathbb{C})$ over $\mathbb{C}$	$\{A^{n \times m}_{ij} \mid A \in M_{n\times m}(\mathbb{C})$ whose entries are zero except for the ij^{th} entry, which is one.$\}$	$n \times m$
$\mathbb{R}[x]$	$\{1, x, x^2, x^3, \dots, x^n, \dots\}$	∞
$\mathbb{R}$ over $\mathbb{Q}$	?	∞

Steinitz with a basis



Proposition Let V be an n-dimensional vector space & let S be a subset of V with k elements. If S is linearly independent then:

a) $k \leq n$

b) If $k=n$ then S is a basis for V

c) There exists a basis B for V such that $S \subseteq B$.

Proof

a) If S is a lin. ind subset of a v.s. V with dimension n $\Rightarrow$ $\exists\ B = \{\underline{v}_1, \ldots, \underline{v}_n\}$ with B lin. ind. & $sp(B) = V$. Steinitz Exchange Lemma $\Rightarrow k \leq n$

b) If $|S| = n$. The S.E.L $\Rightarrow \exists R \subseteq B$ s.t. $|R| = n-k = n-n = 0$ & $sp(S \cup R) = V$. $|R| = 0 \Rightarrow R = \emptyset$ & $sp(S \cup \emptyset) = V = sp(S)$ so S spans V & is linearly independent $\therefore S$ is a basis for V.

c) We have a basis B for V with n elements & S lin. ind. with k elements. S.E.L to find $R \subseteq B$ with $n-k$ elements & $sp(S \cup R) = V$. $|S \cup R| = k + n - k = n.$ * $\Rightarrow S \cup R$ is a basis.

If A spans V & $|A| = n$ then A contains a linearly independent set B with $sp(B) = V$.

Proof by induction on $|A|$

<u>Proposition</u> Let V be a vector space & let U, W be finite dimensional subspaces of V. The following are true:

a) $\dim(U) \leq \dim(V)$ with equality iff $U = V$

b) $\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$

<u>Proof</u>

$\dim(U) < \infty$ by assumption

If $\dim(V) = \infty$ then $\dim(V) \geq \dim(U)$

If $\dim(V) < \infty$ (ie $\dim(V) = n$) then there exists a basis B with n elements. U is a subspace $\Rightarrow U$ is a V.S. U has a basis $B' = \{\underline{v}_1, \dots, \underline{v}_k\}$ which is lin. ind. We apply S.E.L. to get $k \leq n$.

If $\mathbb{R}$ as a vector space over $\mathbb{Q}$ is finite dimensional $\exists \{r_1, \dots, r_n\} = B$ s.t. every real number ~~$\in A$~~ is a linear combination of elements in B.