277

24/12

<u>Prop:</u> If $\underline{u} = c_1\underline{b}_1 + \dots + c_n\underline{b}_n$ = $\underline{u} = c_1'\underline{b}_1' + \dots + c_k'\underline{b}_k'$ ✱ for $\underline{u} \in V$ & $\underline{b}_1, \dots, \underline{b}_n, \underline{b}_1', \dots, \underline{b}_k'$ elements of a basis B then $\{\underline{b}_1, \dots, \underline{b}_n\} = \{\underline{b}_1', \dots, \underline{b}_k'\}$

<u>Def:</u> If $U \subseteq V$ where V is a vector space we say U spans V when ~~the~~ $sp(U) = V$

<u>Prop. proof</u> from last time

If $c_1\underline{b}_1 + \dots + c_n\underline{b}_n = c_1'\underline{b}_1 + \dots + c_n'\underline{b}_n$

then $(c_1 - c_1')\underline{b}_1 + \dots + (c_n - c_n')\underline{b}_n = \underline{0}$

$\Rightarrow c_i - c_i' = 0 \quad \forall i$ because $\underline{b}$ is linearly independent.

<u>Proof</u> (✱) implies $c_1'\underline{b}_1' = c_1\underline{b}_1 + \dots + c_n\underline{b}_n - c_2'\underline{b}_2' - \dots - c_k'\underline{b}_k'$

$\Rightarrow \underline{b}_1' = \frac{1}{c_1'}(c_1\underline{b}_1 + \dots + c_n\underline{b}_n - c_2'\underline{b}_2' - \dots - c_k'\underline{b}_k')$

ie. B is linearly dependent unless $\underline{b}_1' \in \{\underline{b}_1, \dots, \underline{b}_n\}$

Similarly, $\underline{b}_i' \in \{\underline{b}_1, \dots, \underline{b}_n\}$

ie. $\{\underline{b}_1', \dots, \underline{b}_k'\} \subseteq \{\underline{b}_1, \dots, \underline{b}_n\}$

Arguing in the same way,

$\{\underline{b}_1, \dots, \underline{b}_n\} \subseteq \{\underline{b}_1', \dots, \underline{b}_k'\}$

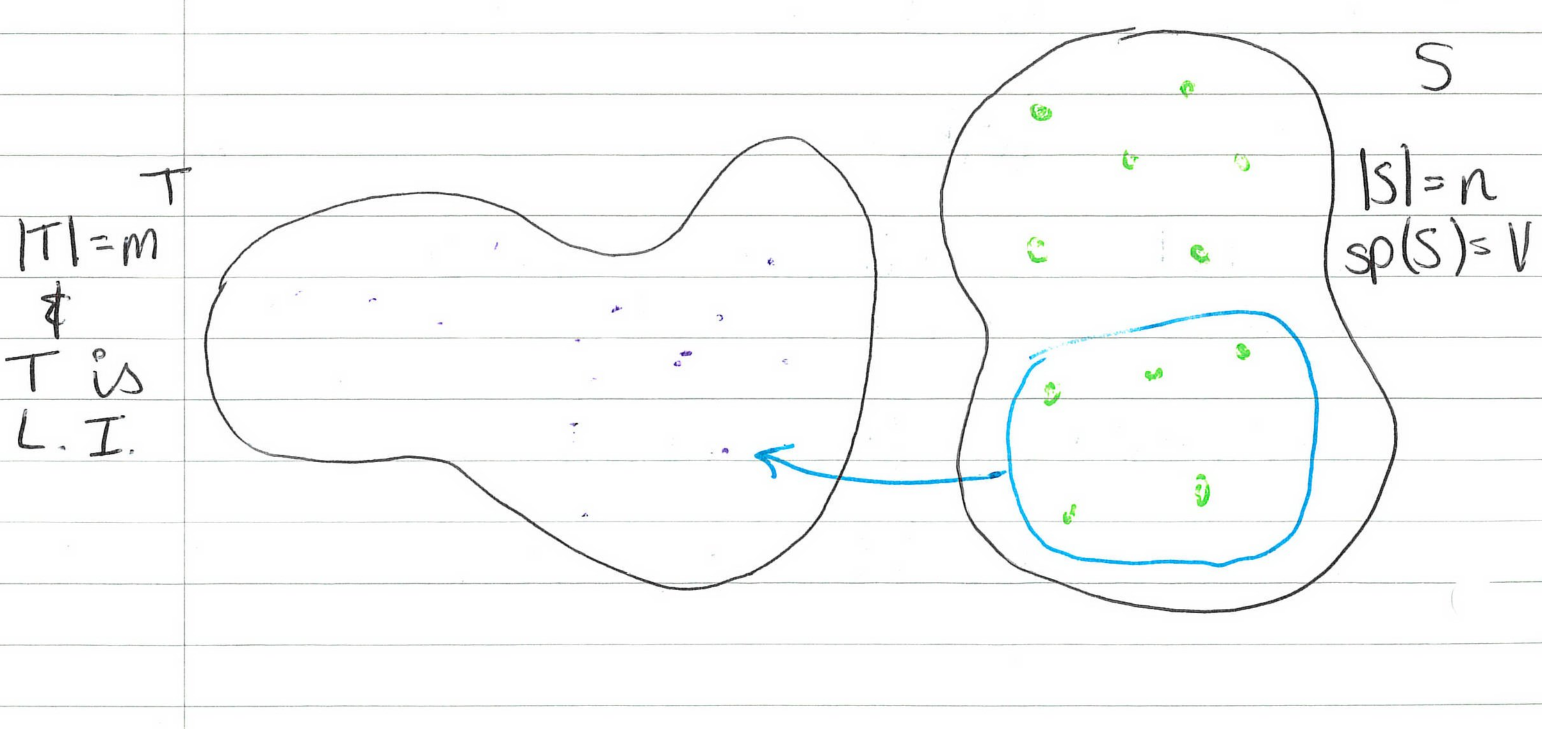
$\underline{Ex}$: $\{1, x, x^2\}$, $\{1, 1+x, 1+x^2\}$,

$\{1, 1+x, 1+x+x^2\}$ are bases for $\mathbb{R}[x]_{\leq 2}$

$\{1, x, x^2, \dots\}$, $\{1, 1+x, 1+x^2, \dots\}$,

$\{1, 1+x, 1+x+x^2, \dots\}$ are bases for $\mathbb{R}[x]$

(Steinitz exchange lemma) Let V be a vector space & let S be a set with n elements such $sp(S) = V$. If T is a linearly independent subset of V with m elements then $m \leq n$ & there exists an $R \subseteq S$ containing $n-m$ elements such that $sp(T \cup R) = V$



Lemma 1.5

- $m \leq n$
- $\exists R \subseteq S$ s.t. $|R| = n - m$
- $sp(T \cup R) = sp(S) = V$

Theorem If B & B' are bases for a vector space V ~~has~~ & B has $n \in \mathbb{N}$ elements then B' has n elements.

Proof B is a basis so $sp(B) = V$. By assumption, $|B| = n$.

If B' is a basis for V then B' is linearly independent. If $|B'| > |B|$ then $\exists A \subseteq B'$ with $|A| = m = n+1$

$\Rightarrow m = n+1 \leq n$ $\quad \#$ $\quad$ ie. $|B'| \leq |B|$

$\uparrow$ SΣL

ie. $|B'| = \cancel{m}\, k \leq n = |B|$. (Apply SΣL)

we gen $n \leq \cancel{m}\, k$.

ie. $|B'| \geq |B|$

Def A vector space V is called finite dimensional with dimension n if there exists a basis B for V with n elements. In this case we write $\dim V = n$. A space that is not finite dimensional is said to be infinite dimensional.

<u>Proof of S.E.L.</u> We prove this by induction on m.

<u>Base Case</u> $m=0$ we set $R=S$

In this case, $m \leq n$

$R \subseteq S$ & $sp(R) = sp(S)$

$|R| = |S| = n = n-m$ $(m=0)$

Assume ** holds for m

Suppose $|T| = m+1$ ie. $T = \{\underline{v}_1, \ldots, \underline{v}_{m+1}\}$

Set. $T' = \{\underline{v}_1, \ldots, \underline{v}_m\}$

by ** $\Rightarrow \exists R' = \{\underline{u}_1, \ldots, \underline{u}_{n-m}\}$ s.t.

$sp(T' \cup R') = sp(S) = V$

$\cancel{\Rightarrow}$ $n-m>0$ because otherwise, $sp(T') = V \Rightarrow T$ is linearly dependent.

ie. $\boxed{n \geq m+1}$

Set $R = \{\underline{u}_2, \ldots, \underline{u}_{n-m}\}$. $\boxed{\begin{aligned}|R| &= n-m-1 \\ &= n-(m+1)\end{aligned}}$

$sp(T' \cup R') = V$ in particular $\underline{v}_{m+1} =$
$a_1 \underline{v}_1 + \ldots + a_1 \underline{v}_m + b_1 \underline{u}_1 + \ldots + b_1 \underline{u}_{n-m}$

for some $a_i, b_j \in \mathbb{F}$

$$\Rightarrow b_1 \underline{u}_1 = a_1 \underline{v}_1 + \dots + a_m \underline{v}_m - \underline{v}_{m+1}$$

$$+ b_2 \underline{u}_2 + \dots + b_{n-m} \underline{u}_{n-m}$$

$$\Rightarrow \underline{u}_1 = \frac{1}{b_1}\Big(a_1 \underline{v}_1 + \dots + a_m \underline{v}_m - \underline{v}_{m+1}$$

$$+ b_2 \underline{u}_2 + \dots + b_{n-m} \underline{u}_{n-m} \Big)$$

$$\boxed{\text{ie. } sp(T \cup R) \underset{\underline{u}_1 \in sp(T \cup R)}{=} sp(T' \cup R') \underset{**}{=} sp(S) = V}$$

$$\Rightarrow T' \cup R' \subseteq sp(T \cup R)$$