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20/2

Homework: Q4, 8, 24 for Tuesday

$C^\infty[0,1] \subseteq C[0,1] \subseteq \mathcal{F}([0,1], \mathbb{R})$

Hint for (d): subspace: verify:

- $\underline{0} \in U+W$
- $z_1, z_2 \in U+W$, $c \in F$ $\quad z_1 + cz_2 \in U+W$

$\underline{Ex}$ $T: \{A \in M_{n\times n}(\mathbb{R}) \mid tr(A) = 0\}$

trace = sum of diagonals

T is a subspace of $M_{n\times n}(\mathbb{R})$ as:

- $\underline{0} \in T$ $\quad \begin{pmatrix} 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & & \end{pmatrix}$
- $A = (a_{ij})$ $\quad B = (b_{ij})$ & $c \in \mathbb{R}$ $\quad A, B \in T$

then $tr(\underbrace{A+cB}_{a_{ij}+cb_{ij}}) = a_{11} + cb_{11} + \dots + a_{nn} + cb_{nn} = 0$

$= (a_{11} + \dots + a_{nn}) + c(b_{11} + \dots + b_{nn}) = 0 + c \cdot 0 = 0$

Hence $T \subseteq M_{n\times n}(\mathbb{R})$ is a subspace.

$\underline{\text{Def}}$ Let V be a vector space over a field $\mathbb{F}$. A set $U \subseteq V$ is said to be linearly dependent if there exists $u_1, \dots, u_n \in U$ & $c_1, \dots, c_n \in \mathbb{F} \setminus \{0\}$ such that $c_1 u_1 + \dots + c_n u_n = \underline{0}$. A set $W \subseteq V$ which is not linearly dependent is said to be linearly independent.

$\underline{Ex}$: $\left\{ \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \subseteq M_{2\times 2}(\mathbb{R})$ (over $\mathbb{R}$)

If A is linearly dependent then there exists $a, b, c \in \mathbb{R}$ such that

$$a\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 2a+b & a+c \\ b+c & a+2b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$a = -c$ $\qquad$ $\cancel{b = c}$ $c = -b \Rightarrow a = b$ ②

$2a + b = 0$ $\qquad$ $a + 2b = 0$ ①

$3a = 3b = 0 \Rightarrow a, b, c = \underline{\underline{0}}$.

Hence, this set is linearly independent.

$\underline{Ex}$ Show $\{ 1+x+x^2, x^2+1, x^2, x-1 \} \subseteq \mathbb{R}[x]$ is linearly dependent.

$\underline{Def}$ Let V be a vector space over a field $\mathbb{F}$, & let $U \subseteq V$. We define

$$sp(U) := \{ \underline{u} = c_1\underline{u}_1 + \ldots + c_k\underline{u}_k \mid \underline{u}_1, \ldots, \underline{u}_k \in U, c_1, \ldots, c_k \in \mathbb{F} \}$$

The set $sp(U)$ is called the span of U.

Ex $\{1, x, \dots, x^n\} \subseteq \mathbb{R}[x]$

then $sp\{1, x, \dots, x^n\} = \mathbb{R}[x]_{\le n}$ polynomials of order $\le n$

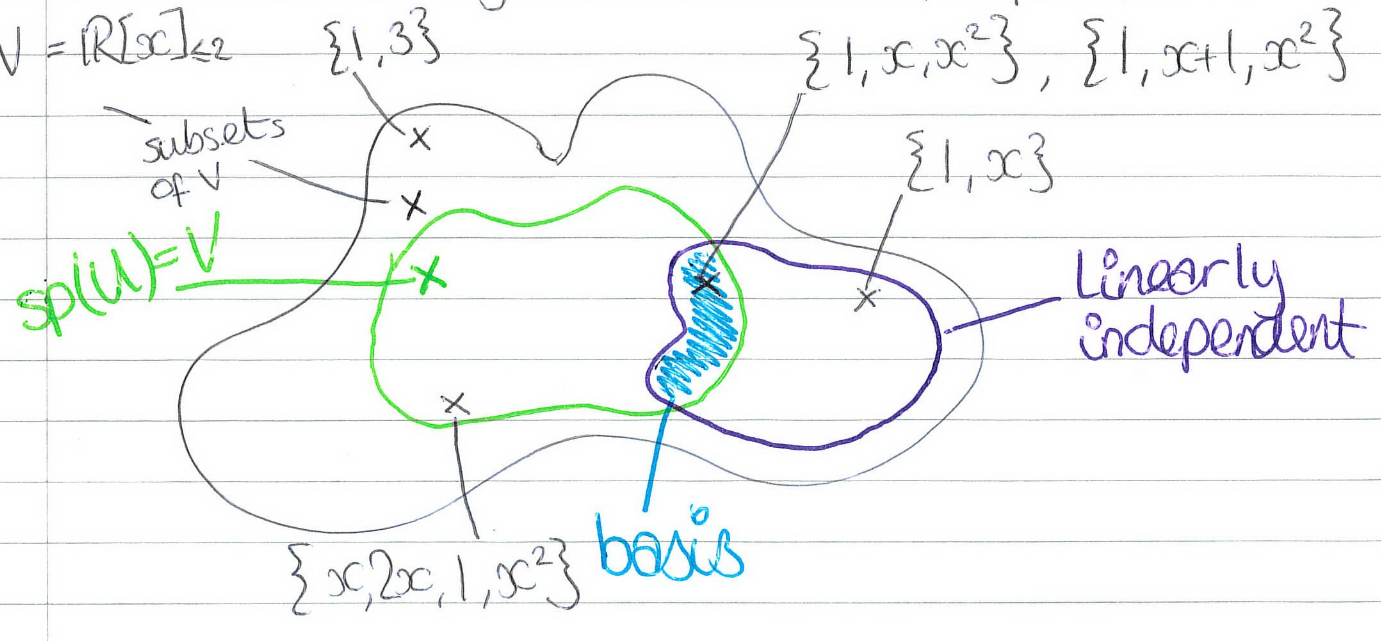
Remark: If $U \subseteq V$ ($\neq \emptyset$; vector space) then $sp(U)$ is a subspace of V. To see this note $\underline{0} = 0\underline{u}$ for $\underline{u} \in U \neq \emptyset$ & $\underline{u} + c\underline{v} \in sp(U)$ when $\underline{u}, \underline{v} \in U$ & $c \in \mathbb{F}$.

Suppose $\underline{u} = c_1\underline{u}_1 + \dots + c_k\underline{u}_k + a_1\underline{w}_1 + \dots + a_n\underline{w}_n$ for some $\underline{u}_i, \underline{w}_i \in U$ & $c_i, a_i \in \mathbb{F}$ & $\underline{v} = b_1\underline{v}_1 + \dots + b_m\underline{v}_m + d_1\underline{w}_1 + \dots + d_n\underline{w}_n$ for some $\underline{v}_i, \underline{x}_i \in V$ & $d_i, b_i \in \mathbb{F}$

$\underline{u} + e\underline{v} = c_1\underline{u}_1 + \dots + c_k\underline{u}_k + (a_1 + ed_1)\underline{w}_1 + (a_n + ed_n)\underline{w}_n + eb_1\underline{v}_1 + \dots + eb_m\underline{v}_m \in Sp(U)$ because $\underline{u}_i, \underline{w}_i, \underline{v}_i \in U$ & $c_i, a_i + ed_i, eb_i \in \mathbb{F}$

Def A set U contained in a vector space V is called a basis for V if U is linearly independent & $sp(U) = V$.

e.g. $V = \mathbb{R}[x]_{\le 2}$



<u>Proposition</u> Let B be a basis for a vector space V over a field $\mathbb{F}$. If $\underline{b}_1, \ldots, b_n \in B$, $c_1, \ldots, c_n, c_1', \ldots, c_n' \in \mathbb{F}$ are such that

$$c_1\underline{b}_1 + \ldots + c_n\underline{b}_n = c_1'\underline{b}_1 + \ldots + c_k'\underline{b}_k$$

then $\cancel{c_n = c_n'}$ $c_i = c_i'$

<u>Proof</u> Suppose B is a basis for V &

$$\underline{v} = c_1\underline{b}_1 + \ldots + c_n\underline{b}_n = c_1'\underline{b}_1' + \ldots + c_k'\underline{b}_k'$$

$$\Rightarrow (c_1\underline{b}_1 + \ldots + c_n\underline{b}_n) - (c_1'\underline{b}_1' + \ldots + c_k'\underline{b}_k') = 0$$

$\Rightarrow$ coefficients of the $\underline{b}_i$, $\underline{b}_i'$ are zero because B is a basis & therefore linearly independent.