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Prop 1.1 $\forall a,b,c \in \mathbb{F}$

a) $a+b=c+b \Rightarrow a=c$
d ~~b~~) $(-1).a=-a$
c) $a.0=0$
e ~~d~~) $(-a).b=a.(-b)=-(a.b)$

Proof of d

$0=0.a$ from c by commutativity ⓚ

$0=(1-1).a$ by inverses F1

$=1.a+(-1).a$ by F3

~~$(-1).a+1.a$~~ $(-1)a+0=\cancel{-a+a}+(-a)$

$-a+0=0+(-1)a=0+-a$

$\overset{(a)}{\Rightarrow} -a=(-1)a$

Definition A vector space V over a field $\mathbb{F}$ is a set with a binary operation $+: V\times V\to V$ (called addition) & a function $\cdot: \mathbb{F}\times V \to V$ (called scalar multiplication)

VS1 V is an Abelian group w.r.t. $+$

VS2 For every $\underline{v} \in V$ we have $1\underline{v} = \underline{v}$

VS3 for all $a, b \in \mathbb{F}$ & $\underline{v} \in V$ we have
$(ab)v = a(bv)$

VS4 for all $a \in \mathbb{F}$ & $\underline{u}, \underline{v} \in V$ we have
$a(\underline{u} + \underline{v}) = a\underline{u} + a\underline{v}$

VS5 for all $a, b \in \mathbb{F}$ & $\underline{v} \in V$ we have
$(a+b)\underline{v} = a\underline{v} + b\underline{v}$

Examples:

$\mathbb{R}^n$ over $\mathbb{R}$ is a VS

$\mathbb{C}^n$ over $\mathbb{C}$ is a VS

$\mathbb{F}^n$ over $\mathbb{F}$ is a VS over a field $\mathbb{F}$

$\mathbb{F}^n = \{(a_1, \ldots, a_n)^T \mid a_i \in \mathbb{F}\}$

$+$ is defined by $(a_1, \ldots, a_n)^T + (b_1, \ldots, b_n)^T$
$= (a_1 + b_1, \ldots, a_n + b_n)^T$

$\cdot$ is defined by $c(a_1, \ldots, a_n)^T = (ca_1, \ldots, ca_n)^T$

~~Conventions~~:

~~Formally~~

Conventions

Formal	Informal
$+(\underline{u}, \underline{v})$	$\underline{u} + \underline{v}$
$\cdot(a, \underline{v})$	$a . \underline{v}$
id w.r.t. $+$	$\underline{0}$
$\underline{v}$ inv. w.r.t. $+$	$-\underline{v}$
$\underline{u} + (-\underline{v})$	$\underline{u} - \underline{v}$

$\mathcal{F}(X, \mathbb{F}) = \{f : X \to \mathbb{F}\}$ over $\mathbb{F}$ with

$+$ def. by $(f+g) : x \mapsto f(x) + g(x)$

$\cdot$ def. by $cf : x \mapsto cf(x) \quad (c \in \mathbb{F})$

Claim this is a vector space.

Proof:

$\mathcal{F}(X, \mathbb{F})$ is an Abelian group under $+$ as $f(x) + 0 . x = f(x)$ $[0 : x \mapsto 0]$
$f(x) - f(x) = 0$, $f(x) + g(x) = (f+g)(x)$
$= g(x) + f(x)$.

$(f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x))$

$\forall x \in \mathbb{F}$ as $\mathbb{F}$ is an Abelian group w.r.t. $+$

VS3 & VS4 & VS5 are similar

Proposition 1.2

The following statements are true for any vector space V over $\mathbb{F}$

a) $0\underline{v} = \underline{0} \quad \forall \underline{v} \in V$

b) $(-a)\underline{v} = -(a\underline{v}) = a(-\underline{v}) \quad \forall a \in \mathbb{F} \; \& \; \underline{v} \in V$

c) $a\underline{0} = \underline{0} \quad \forall a \in \mathbb{F}$

Proof

(a) $\underline{0} + 0\underline{v} \overset{\text{id. VS1}}{=} 0\underline{v} = (0+0)\underline{v}$

$\underset{\text{distr.}}{=} 0\underline{v} + 0\underline{v} \underset{\text{same as prop 1.1}}{\Rightarrow} \underline{0} = 0\underline{v}$

(b) $(-a)\underline{v} = (-1.a)\underline{v} \underset{\text{comm.}}{=} (a.-1)\underline{v} \underset{\text{VS3}}{=} a.(-1.\underline{v})$

$\overset{\text{claim this works}}{=} a.(-v)$

(c) $\underline{0} + a\underline{0} = a\underline{0} = a(\underline{0}+\underline{0}) = a\underline{0} + a\underline{0}$

$\overset{\text{cancelling}}{\Rightarrow} \underline{0} = a\underline{0}$

$C[a,b] := \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}$

$C[a,b] \subseteq \mathcal{F}([a,b], \mathbb{R})$

Definition

Let V be a vector space over a field $\mathbb{F}$. A set $W \subseteq V$ is called a subspace of V if the following hold:

* $\underline{0} \in W$
* $\underline{u} + \underline{v} \in W \quad \forall\ \underline{u}, \underline{v} \in W$
* $c\underline{v} \in W \quad \forall\ c \in \mathbb{F}\ \&\ \underline{v} \in W$

Proof

$\underline{0} : x \mapsto 0$ is in $C[a,b]$

$f+g$ is continuous whenever f & g are.

Similarly cf is continuous when f is.

Ex

$M_{m \times n}(\mathbb{F})$ is the set of $m \times n$ matrices with entries in $\mathbb{F}$. It is a vector space.

Consider,

$A = \{B \in M_{2\times 2}(\mathbb{R}) \mid \det(B) = 1\}$

The identity $\underline{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is not an element of A. So, A is not a subspace of $M_{2\times2}(\mathbb{R})$