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MAS277 - Lecture 19

Announcements * Thanks

* Website ~~will~~ be updated soon

* Dr James Cranch from next week

*** Typo alert - Def 19 (pg 21) IP2

$$\langle c\underline{u}, \underline{v}\rangle = c\langle \underline{u},\underline{v}\rangle$$

- Def 27 (pg 27)

$$S^{\perp} := \{\underline{v}\in V : \langle \underline{v},\underline{u}\rangle = 0 \text{ for all } \underline{u}\in S\}$$

- Thm 4.3 (pg 29)

$$\mathrm{Proj}_U(\underline{v}) = \sum \frac{\langle \underline{v},\underline{u}_i\rangle}{\langle \underline{u}_i,\underline{u}_i\rangle}\underline{u}_i$$

Chapter 3 Revision

* Inner product spaces

Exam style q: Define an inner product space.

Bad answer:
1. $\langle \underline{u}+\underline{v},\underline{w}\rangle = \langle \underline{u},\underline{w}\rangle + \langle \underline{v},\underline{w}\rangle$
2. $\langle c\underline{u},\underline{v}\rangle = c\langle \underline{u},\underline{v}\rangle$
3. $\langle \underline{u},\underline{v}\rangle = \overline{\langle \underline{v},\underline{u}\rangle}$
4. $\langle \underline{u},\underline{u}\rangle > 0$.

Good Answer: An inner product space is a vector space V over a field $\mathbb{F}\subseteq\mathbb{C}$ together with a map $\langle\,,\rangle : V\times V\to\mathbb{F}$ such that the $\underline{\left(\underline{u},\underline{v}\right)\longmapsto\langle\underline{u},\underline{v}\rangle}$ following properties hold:

① For all $\underline{u},\underline{v},\underline{w}\in V$
$$\langle \underline{u}+\underline{v},\underline{w}\rangle = \langle \underline{u},\underline{w}\rangle + \langle \underline{v},\underline{w}\rangle.$$
② For all $\underline{u},\underline{v}\in V$ & $c\in\mathbb{F}$ $\langle c\underline{u},\underline{v}\rangle = c\langle \underline{u},\underline{v}\rangle$

③ For all $\underline{u},\underline{v}\in V$ $\langle \underline{u},\underline{v}\rangle = \overline{\langle \underline{v},\underline{u}\rangle}$

④ For all $\underline{u}\in V\setminus\{\underline{0}\}$ $\langle \underline{u},\underline{u}\rangle\in\mathbb{R}^{+}$

Standard example of inner product spaces:

- $\mathbb{R}^n$, $\langle \underline{x}, \underline{y} \rangle = \underline{x} \cdot \underline{y}$
- $\mathbb{C}^n$, $\langle \underline{x}, \underline{y} \rangle = \sum_{i=1}^{n} x_i \bar{y}_i$ where $\underline{x} = (x_1, x_2, \dots, x_n)$
  $\underline{y} = (y_1, y_2, \dots, y_n)$
- $M_{n \times n}(\mathbb{R})$, $\langle A, B \rangle = \text{trace}(AB^T)$
- $C[a,b]$, $\langle f(x), g(x) \rangle = \int_a^b f(x) g(x)\, dx$

Excercise: verify that each of the ~~following~~ above is an inner product.

* Norms

In $\mathbb{R}^n$, $\|\underline{x}\| := \sqrt{\underline{x} \cdot \underline{x}}$

In $(V, \langle , \rangle)$, $\|\underline{v}\| := \sqrt{\langle \underline{v}, \underline{v} \rangle}$

e.g. in $(C[-1,1], \langle , \rangle)$

$f(x) = x \longrightarrow$ $\|x\| := \sqrt{\langle x, x \rangle} := \sqrt{\int_{-1}^{1} (x)(x)\, dx}$

$= \sqrt{\int_{-1}^{1} x^2\, dx} = \sqrt{2/3}$

Def: $\underline{u}, \underline{v} \in V$ are orthogonal if $\langle \underline{u}, \underline{v} \rangle = 0$.

Exam style q: Prove that in an I.P.S.
$\|\underline{x} + \underline{y}\|^2 + \|\underline{x} - \underline{y}\|^2 = 2\|\underline{x}\|^2 + 2\|\underline{y}\|^2$.

Good Answer: We have $\|\underline{v}\| := \sqrt{\langle \underline{v}, \underline{v} \rangle}$ in an inner product space.

So $\|\underline{x} + \underline{y}\|^2 + \|\underline{x} - \underline{y}\|^2 = \langle \underline{x} + \underline{y}, \underline{x} + \underline{y} \rangle + \langle \underline{x} - \underline{y}, \underline{x} - \underline{y} \rangle$

By linearity $\Rightarrow = \langle x, x \rangle + \cancel{\langle x, y \rangle} + \cancel{\langle y, x \rangle} + \langle y, y \rangle$
$+ \langle x, x \rangle - \cancel{\langle x, y \rangle} - \cancel{\langle y, x \rangle} + \langle y, y \rangle$

$= 2\langle x, x \rangle + 2\langle y, y \rangle = 2\|x\|^2 + 2\|y\|^2$.

$\square$

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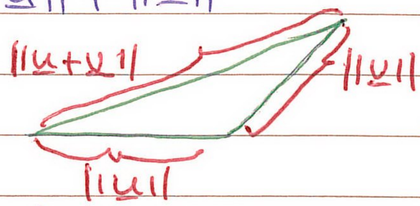
* Cauchy-Schwarz and triangle inequality

Triangle inequality: $\|\underline{u}+\underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$



C-S inequality: $|\langle \underline{u}, \underline{v} \rangle| \leq \|\underline{u}\| \cdot \|\underline{v}\|$

Exam style q: Show that

$$\int_0^{\pi} \sin(x) f(x)\,dx \leq \sqrt{\frac{\pi}{2}} \sqrt{\int_0^{\pi} (f(x))^2\,dx}$$

When $f(x) \geq 0$ on $[0, \pi]$.

Quick Answer In $C[0,\pi]$

$$|\langle \sin x, f(x) \rangle| := \left| \int_0^{\pi} \sin x\, f(x)\,dx \right|$$

as $\sin(x), f(x) \geq 0$ on $[0,\pi]$

$$= \int_0^{\pi} \sin x\, f(x)\,dx \leq \|\sin x\| \cdot \|f(x)\|$$

by C.S.

$$= \sqrt{\frac{\pi}{2}} \sqrt{\int_0^{\pi} (f(x))^2\,dx}$$

$\sqrt{\int_0^{\pi} \sin^2(x)\,dx}$

Ex: Check the constant

* Adjoint

$L: V \to V$, (if) there exist a map (Def) $L^*: V \to V$ such that $\langle L(\underline{u}), \underline{v} \rangle = \langle \underline{u}, L^*(\underline{v}) \rangle$.

- Be able to play with definition of adjoint.

Exam style q. Show $V = \{f(x) \in C[0,1] : f(0) = f(1) = 0\}$ is a vector space.

Solution: As $V \subseteq C[0,1]$ we only need to show V is a subspace.

$D : f(x) \mapsto f'(x)$ has an adjoint

$\langle D(f(x)), g(x) \rangle = \int_0^1 f'(x) g(x)\, dx = \dots = \int_0^1 f(x)(-g'(x))\, dx$

i.e. $D^* : h(x) \mapsto -h'(x)$.