28/4

* Q1, 2, 3, 7, 8, 9 due tomorrow
* Surgery day Wednesday until 3
* Weeks 10 - 12 cover (Back 3/6)
* Everything available by end of week on website
* Chp 3 revision session on Thursday

<u>Proposition</u> All eigenvalues of a self-adjoint linear operator are real.

<u>Proof</u> Suppose λ is an eigenvalue of a linear map $L: V \to V$ & $\underline{v}$ its corresponding eigenvector.

$$\langle L(\underline{v}), \underline{v} \rangle = \langle \lambda \underline{v}, \underline{v} \rangle = \bar{\lambda} \langle \underline{v}, \underline{v} \rangle$$

$$\Vert$$

$$\langle \underline{v}, L^*(\underline{v}) \rangle \overset{L = L^* \text{ as } L \text{ is self-adjoint}}{=} \langle \underline{v}, L(\underline{v}) \rangle = \langle \underline{v}, \lambda \underline{v} \rangle$$

$$= \overline{\langle \lambda \underline{v}, \underline{v} \rangle} = \overline{\bar{\lambda} \langle \underline{v}, \underline{v} \rangle} = \lambda \overline{\langle \underline{v}, \underline{v} \rangle}$$

$\langle \underline{v}, \underline{v} \rangle \in \mathbb{R}$

$$= \lambda \langle \underline{v}, \underline{v} \rangle$$

ie $\bar{\lambda} \langle \underline{v}, \underline{v} \rangle = \lambda \langle \underline{v}, \underline{v} \rangle$ & $\langle \underline{v}, \underline{v} \rangle \neq 0$

$$\Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}$$

* If V is finite dimensional & $L: V \to V$ is linear then L is "like" a matrix A_L
* $(V, \langle\ ,\ \rangle)$ is "like" $(\mathbb{R}^n, \cdot)$ or $(\mathbb{C}^n, \cdot)$
* If $(V, \langle\ ,\ \rangle)$ is "like" $(\mathbb{R}^n, \cdot)$ then $L^*: V \to V$ is "like" $(A_L)^T$

* If $(V, \langle\, , \,\rangle)$ is "like" $(\mathbb{C}^n, \cdot)$ then $L^*: V \to V$ is "like" $(A_L)^T$

Some ch4 Definitions

Let V be a real inner product space. We make the following definitions:

* A set $S \subset V$ is said to be orthogonal if $\langle \underline{u}, \underline{v} \rangle = 0$ for every pair of distinct values $\underline{u}, \underline{v} \in S$.
* A vector $\underline{v} \in V$ is a unit if $\|\underline{v}\| = 1$
* A set $S \subset V$ is said to be orthonormal if S is orthogonal & each element of S is a unit.
* A set $B \subset V$ is called an orthogonal/orthonormal basis if B is a basis for V & is orthogonal/orthonormal.
* For S ~~to~~ be a subset of an inner product space V. We define
  $S^{\perp} = \{\underline{v} \in V \mid \langle \underline{v}, \underline{u} \rangle \ \forall\ \underline{u} \in S\}$

<u>Ex</u> W is a subspace of V & $W^{\perp}$ is a subspace of V & $W \cap W^{\perp} = \{\underline{0}\}$

<u>Lemma</u> If W is a subspace of an inner product space V & $L: V \to V$ is a self-adjoint operator with $L(W) \subseteq W$ then $L(W^{\perp}) \subseteq W^{\perp}$.

Theorem If $L: V \to V$ is a self-adjoint operator on an n-dimensional vector space then there exists an orthonormal basis $B = \{\underline{v}_1, \dots, \underline{v}_n\}$ with each $\underline{v}_i$ an eigenvector of L.

Pf of Lemma suppose ~~If~~ $L: V \to V$ is s-a & $L(W) \subseteq W$. ~~then for all $\underline{v} \in W^\perp$~~

If $\underline{v} \in W^\perp$ then for all $\underline{w} \in W$

$$\langle L(\underline{v}), \underline{w} \rangle \overset{\text{self-adjoint}}{=} \langle \underline{v}, L(\underline{w}) \rangle$$

$$= 0 \text{ because } L(\underline{w}) \in W \text{ \& } \underline{v} \in W^\perp$$

ie. $L(\underline{v}) \in W^\perp$.

Pf of Theorem Proof by induction on $\dim(V)$

Base case: $\dim(V) = 1$. Let $\underline{v} \in V \setminus \{\underline{0}\}$. $\{\underline{v}\}$ basis for V.
We have $L(\underline{v}) = \lambda \underline{v}$ for some $\lambda \in \mathbb{F}$.

ie. $\left\{ \frac{\underline{v}}{\|\underline{v}\|} \right\}$ is an orthonormal basis for V.

Assume orthonormal basis $B = \{\underline{v}_1, \dots, \underline{v}_n\}$ of eigenvectors of L exists, when $\dim(V) = n$.

Suppose $\dim(V) = n+1$. The characteristic polynomial $\chi_L(\lambda) \in \mathbb{C}[\lambda]$ splits.

i.e. $\chi_L(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_{n+1})$

Let $\underline{u}$ be a unit eigenvector for λ_1. Set $W = sp\{\underline{u}\}$

~~So~~ W is a subspace of V with dimension 1.

Hence, $W^\perp$ is a subspace of V with dimension n

$W \cap W^\perp = 0$

i.e. there exists an orthonormal basis $\{\underline{v}_1, \dots, \underline{v}_n\}$ for $W^\perp$ by the inductive hypothesis.

Hence $\{\underline{u}, \underline{v}_1, \dots, \underline{v}_n\}$ is an orthonormal basis for $W \oplus W^\perp = V$.