Sample exam being put on website over next few days.

Suggestion: If you see a statement involving norms, translate into inner products.

Ex: Use the triangle inequality to obtain a bound for

$$\cancel{\sqrt{\int_0^1 (\cos(x)+e^{x})^2\,dx}}$$

$$\sqrt{\int_0^1 (\cos(x)+x^{30})^2\,dx}$$

Note, in $C[0,1]$ with $f(x)=\cos(x)$ & $g(x)=x^{30}$, we have

$$\|f(x)+g(x)\| \leq \|f(x)\| + \|g(x)\|$$

by the triangle inequality.

~~This means~~ As $\langle p(x), q(x)\rangle =$

$\cancel{\|p(x)\| =}$ $\int_0^1 p(x)q(x)\,dx$ in $C[0,1]$

we see $\|f(x)+g(x)\| = \sqrt{\langle f(x)+g(x), f(x)+g(x)\rangle}$

$$= \sqrt{\int_0^1 (\cos(x)+x^{30})^2\,dx}$$

Δ ineq

$$\leq \|f(x)\| + \|g(x)\|$$

$$= \sqrt{\int_0^1 \cos^2(x)\,dx} + \sqrt{\int_0^1 x^{60}\,dx}$$

$$= \frac{1}{\sqrt{61}} + \sqrt{\frac{1}{2}\int_0^1 1+\cos(2x)\,dx}$$

$$= \frac{1}{\sqrt{61}} + \sqrt{\frac{1}{2} + \left[\frac{1}{2}\sin(2x)\right]_0^1}$$

$$= \frac{1}{\sqrt{61}} + \frac{1}{\sqrt{2}} + \sqrt{\frac{\sin(2)}{2}}$$

<u>Theorem</u> Let V be an inner product space over a field $\mathbb{F}$ & let $L: V \to V$ with $\langle L(\underline{u}), \underline{v} \rangle = \langle \underline{u}, L^*(\underline{v}) \rangle$ for all $\underline{u}, \underline{v} \in V$, then L^* is linear & unique. Furthermore, if V is finite dimensional then L^* exists.

<u>Motivation:</u> $\langle \underline{x}, \underline{y} \rangle := \underline{x} \cdot \underline{y}$ $\qquad \underline{\mathbb{R}^n}$

Consider $\underbrace{A}_{n\times n \text{ matrix}}\underline{x} \cdot \underline{y}$

$$= \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^n a_{1i} x_i \\ \vdots \\ \sum_{i=1}^n a_{ni} x_i \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$= \sum_{i=1}^n a_{1i} x_i y_1 + \ldots + \sum_{i=1}^n a_{ni} x_i y_n$$

$$= (a_{11} y_1 + \ldots + a_{n1} y_n) x_1 + \ldots$$

$$+ (a_{1n} y_1 + \ldots + a_{nn} y_n) x_n$$

$=$

$$= \begin{pmatrix} a_{11} y_1 + \dots + a_{n1} y_n \\ \vdots \\ a_{1n} y_1 + \dots + a_{nn} y_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= A^T \underline{y} \cdot \underline{x} = \underline{x} \cdot A^T \underline{y}$$

$$= \langle \underline{x}, A^T \underline{y} \rangle$$

In $\mathbb{R}^n$, if $A = A^T$

$$\Rightarrow \overset{\text{eigenvalues}}{D} = Q^T A Q$$

Proof If L^* exists to show L^* is linear we must verify $L^*(\underline{x} + c\underline{y}) = L^*(\underline{x}) + cL^*(\underline{y})$.

Suppose $\langle \underline{x}, L^*(\underline{x} + c\underline{y}) \rangle$

Thm 3.7

$$= \langle L(\underline{u}), \underline{x} + c\underline{y} \rangle$$

$$= \langle L(\underline{u}), \underline{x} \rangle + c \langle L(\underline{u}), \underline{y} \rangle$$

$$= \langle \underline{u}, L^*(\underline{x}) \rangle + \langle \underline{u}, cL^*(\underline{y}) \rangle$$

$$= \langle \underline{u}, L^*(\underline{x}) + cL^*(\underline{y}) \rangle \quad \forall \underline{u}$$

Prop 3.1(4) $\langle \underline{x}, \underline{u} \rangle = \langle \underline{x}, \underline{v} \rangle \ \forall \underline{x} \Rightarrow \underline{u} = \underline{v}$

$$\Rightarrow L^*(\underline{x} + c\underline{y}) = L^*(\underline{x}) + cL^*(\underline{y})$$

Uniqueness: Suppose $\langle L(x), \underline{y} \rangle = \langle \underline{x}, T(\underline{y}) \rangle$

$$= \langle x, l(\underline{y}) \rangle \quad \forall \underline{x}$$

$$\Rightarrow \langle \underline{x}, T(\underline{y}) \rangle - \langle \underline{x}, l(\underline{y}) \rangle = 0$$

ie, $\langle \underline{x}, T(y) - l(y) \rangle = 0 \quad \forall \underline{x}$

but $\langle \underline{x}, \underline{0} \rangle = \langle \underline{x}, 0.\underline{u} \rangle = 0 \langle \underline{x}, \underline{u} \rangle = 0$

ie. $T(y) - l(y) = \underline{0}$ by 3.i(4)

<u>Existence</u> (idea)
$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\varphi_A"] \arrow[d, leftrightarrow] & \mathbb{R}^n \arrow[d, leftrightarrow] \\ V \arrow[r, "{[\varphi_A]_B^C}"] & V \end{tikzcd}$$

$$\rightsquigarrow \quad \begin{tikzcd} \mathbb{R}^n \arrow[r, "\varphi_A^*"] \arrow[d, leftrightarrow] & \mathbb{R}^n \arrow[d, leftrightarrow] \\ V \arrow[r, "{([\varphi_A]_B^C)^T}"] & V \end{tikzcd}$$

<u>Def</u> If the linear transformation L^* from Thm 3.7 exists then it is called the adjoint of L.

<u>Prop</u> Let V be an inner product space & Let L, N be linear operators of V. If L^* & T^* exist then the following hold:

a) $(L+T)^* = L^* + T^*$

b) $(cL)^* = cL^*$

Proof exercise

c) $(LT)^* = T^* L^*$

d) $(L^*)^* = L$

<u>Proof of (a)</u>

$\langle (L+T)\underline{x}, \underline{y} \rangle = \langle L(\underline{x}) + T(\underline{x}), \underline{y} \rangle$
$= \langle L(\underline{x}), \underline{y} \rangle + \langle T(\underline{x}), \underline{y} \rangle$
$= \langle \underline{x}, L^*(\underline{y}) \rangle + \langle \underline{x}, T^*(\underline{y}) \rangle$
$= \langle \underline{x}, L^* + T^*(\underline{y}) \rangle$

(c) $\langle \underline{x}, (LT)^* \underline{y} = \langle L \circ T(\underline{x}), \underline{y} \rangle = \langle T(\underline{x}), L^*(\underline{y}) \rangle = \langle \underline{x}, T^* L^*(\underline{y}) \rangle$