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Lecture Tomorrow at 2pm, LT 1

Ex: Show that $\int_0^{\frac{\pi}{2}} \sqrt{\cos(x)}\, dx \leq \frac{\pi}{2}$

In $C[a,b]$ we defined an inner product by $\langle f(x), g(x) \rangle = \int_a^b f(x)g(x)\,dx$

We set $f(x) = 1$, $g(x) = \sqrt{\cos(x)}$ so the C-S Inequality tells us

$$|\langle f,g \rangle| \leq \|f(x)\| . \|g(x)\|$$
$$= \sqrt{\langle f,f \rangle} \sqrt{\langle g,g \rangle}$$
$$= \sqrt{\int_0^{\frac{\pi}{2}} 1 \, dx} \sqrt{\int_0^{\frac{\pi}{2}} \cos(x)\, dx}$$
$$= \sqrt{\frac{\pi}{2}}$$

$$|\langle f,g \rangle| = \left| \int_0^{\frac{\pi}{2}} \sqrt{\cos(x)}\, dx \right|$$
$$= \int_0^{\frac{\pi}{2}} \sqrt{\cos(x)}\, dx$$

$\sqrt{\cos(x)} \geq 0 \quad \forall x \in [0, \frac{\pi}{2}]$

Def Two vectors $\underline{u}, \underline{v}$ in an inner product space V are said to be perpendicular if $\langle \underline{u}, \underline{v} \rangle = 0$

Theorem (Pythagoras) $\underline{u}, \underline{v}$ are orthogonal vectors in a real inner product space iff

$$\|\underline{u} + \underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2$$

<u>Proof</u> $\|\underline{u}+\underline{v}\|^2 = \langle \underline{u}+\underline{v}, \underline{u}+\underline{v} \rangle$

$$= \langle \underline{u}, \underline{u}+\underline{v} \rangle + \langle \underline{v}, \underline{u}+\underline{v} \rangle$$

$$= \langle \underline{u}, \underline{u} \rangle + \langle \underline{u}, \underline{v} \rangle + \langle \underline{v}, \underline{u} \rangle + \langle \underline{v}, \underline{v} \rangle$$

$$= \langle \underline{u}, \underline{u} \rangle + 2\langle \underline{u}, \underline{v} \rangle + \langle \underline{v}, \underline{v} \rangle$$

$$= \|\underline{u}\|^2 + \|\underline{v}\|^2 + 2\langle \underline{u}, \underline{v} \rangle$$

$$\Leftrightarrow \langle \underline{u}, \underline{v} \rangle = 0$$

$=0$ if ~~because~~ $\underline{u}, \underline{v}$ perpendicular

<u>Prop</u> (Parallelogram law) Let V be an inner product space. For $u, v \in V$ the following identity holds:

$$\|\underline{u}+\underline{v}\|^2 + \|\underline{u}-\underline{v}\|^2 = 2\|\underline{u}\|^2 + 2\|\underline{v}\|^2$$

<u>Proof</u> $\|\underline{u}+\underline{v}\|^2 + \|\underline{u}-\underline{v}\|^2$

$$= \langle \underline{u}+\underline{v}, \underline{u}+\underline{v} \rangle + \langle \underline{u}-\underline{v}, \underline{u}-\underline{v} \rangle$$

$$= \langle \underline{u}, \underline{u} \rangle + 2\langle \underline{u}, \underline{v} \rangle + \langle \underline{v}, \underline{v} \rangle$$

$$+ \langle \underline{u}, \underline{u} \rangle + 2\langle \underline{u}, -\underline{v} \rangle + \langle -\underline{v}, -\underline{v} \rangle$$ Pull out -1

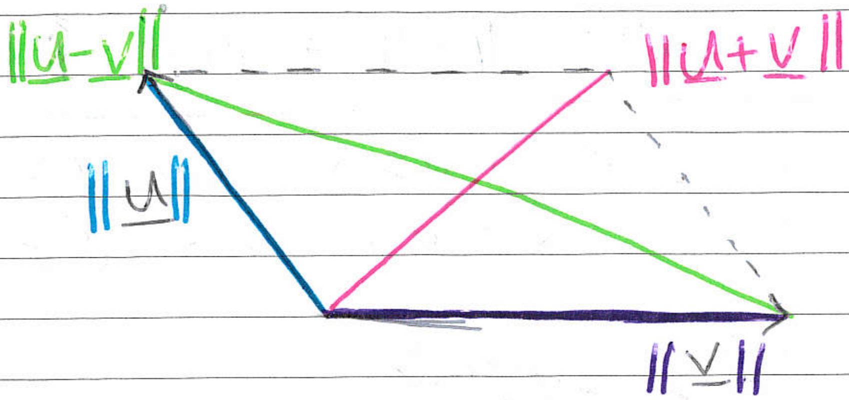
$$= \langle \underline{u}, \underline{u} \rangle + \cancel{2\langle \underline{u}, \underline{v} \rangle} + \langle \underline{v}, \underline{v} \rangle$$

$$+ \langle \underline{u}, \underline{u} \rangle \cancel{-2\langle \underline{u}, \underline{v} \rangle} + \langle \underline{v}, \underline{v} \rangle$$

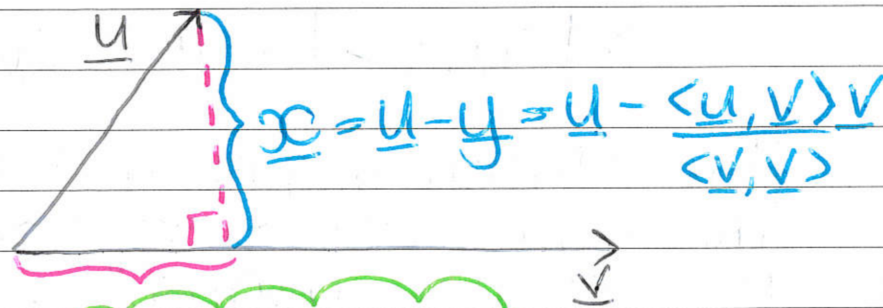
$$= 2\langle \underline{u}, \underline{u} \rangle + 2\langle \underline{v}, \underline{v} \rangle$$

$$= 2\|\underline{u}\|^2 + 2\|\underline{v}\|^2$$

In $\mathbb{R}^2$, Prop 3.5



Proof of Cauchy-Schwarz



set $\boxed{\underline{y} = \frac{\langle \underline{u}, \underline{v} \rangle}{\langle \underline{v}, \underline{v} \rangle}\underline{v}}$

See Ch 4

Note: $\underline{x} + \underline{y} = \underline{u}$

$$\langle \underline{x}, \underline{y} \rangle = \left\langle \underline{u} - \frac{\langle \underline{u}, \underline{v} \rangle}{\langle \underline{v}, \underline{v} \rangle}\underline{v}, \frac{\langle \underline{u}, \underline{v} \rangle}{\langle \underline{v}, \underline{v} \rangle}\underline{v} \right\rangle$$

$$= \frac{\langle \underline{u}, \underline{v} \rangle}{\langle \underline{v}, \underline{v} \rangle} \left\{ \langle \underline{u}, \underline{v} \rangle - \frac{\langle \underline{u}, \underline{v} \rangle^2}{\langle \underline{v}, \underline{v} \rangle^{\cancel{2}}} \cancel{\langle \underline{v}, \underline{v} \rangle} \right.$$

$$= 0$$

We use Pythagoras' Thm to get

$$\|\underline{u}\|^2 = \|\underline{x}+\underline{y}\|^2 = \|\underline{x}\|^2 + \|\underline{y}\|^2 \geq \|\underline{y}\|^2$$

$$= \langle y, y \rangle = \left\langle \frac{\langle \underline{u}, \underline{v} \rangle \underline{v}}{\langle \underline{v}, \underline{v} \rangle}, \frac{\langle \underline{u}, \underline{v} \rangle \underline{v}}{\langle \underline{v}, \underline{v} \rangle} \right\rangle$$

$$= \frac{\langle \underline{u}, \underline{v} \rangle^2}{\langle \underline{v}, \underline{v} \rangle^{\cancel{2}}} \cancel{\langle \underline{v}, \underline{v} \rangle}$$

$$\Rightarrow \|\underline{u}\|^2 \|\underline{v}\|^2 \geq \langle \underline{u}, \underline{v} \rangle^2$$

$$\Rightarrow |\langle \underline{u}, \underline{v} \rangle| \leq \|\underline{u}\| \|\underline{v}\|$$

Ex: Establish equality statement in C-S inequality

Thm (Triangle inequality) Let V be an inner product space. For $\underline{u}, \underline{v} \in V$ the following inequality holds:

$$\|\underline{u}+\underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$$

Proof $\|\underline{u}+\underline{v}\|^2 = \langle \underline{u}+\underline{v}, \underline{u}+\underline{v} \rangle$

$$= \langle \underline{u}, \underline{u} \rangle + 2\langle \underline{u}, \underline{v} \rangle + \langle \underline{v}, \underline{v} \rangle$$

$$= \|\underline{u}\|^2 + 2\langle \underline{u}, \underline{v} \rangle + \|\underline{v}\|^2$$

$$\leq \|\underline{u}\|^2 + 2|\langle \underline{u}, \underline{v} \rangle| + \|\underline{v}\|^2$$

$$\leq \|\underline{u}\|^2 + 2\|\underline{u}\|\|\underline{v}\| + \|\underline{v}\|^2$$

$$= (\|u\| + \|v\|)^2$$ taking $\sqrt{\ }$

$$\Rightarrow \|\underline{u}+\underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$$

Ex: Show Δ

$$\sqrt{\int_0^1 (x^{10}+e^x)^2 \, dx} \leq \frac{1}{\sqrt{21}} + \sqrt{\frac{e-1}{2}}$$