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Homework: Q1, 2, 3, 7, 8, 9

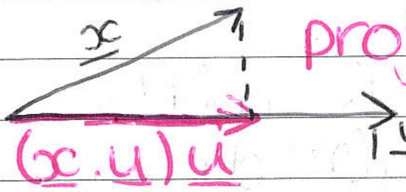
~~Wednesday~~/Friday Lecture
At 2

$\mathbb{R}^n \overset{\text{abstracted}}{\rightsquigarrow}$ Vector spaces

$\underline{x} . \underline{y} \rightsquigarrow$ inner product

length $|\underline{x}| = \sqrt{\underline{x} . \underline{x}}$

angle $\underline{x} . \underline{y} = |\underline{x}| |\underline{y}| \cos \theta$



projections

Def Let V be a vector space over a field $\mathbb{F} \subseteq \mathbb{C}$. An inner product is a map $\langle \, , \, \rangle : V \times V \to \mathbb{F}$ given by $(\underline{u}, \underline{v}) \mapsto \langle \underline{u}, \underline{v} \rangle \in \mathbb{F}$ satisfying the following.

IP1 $\langle \underline{u} + \underline{v}, \underline{w} \rangle = \langle \underline{u}, \underline{w} \rangle + \langle \underline{v}, \underline{w} \rangle \quad \forall \underline{u}, \underline{v}, \underline{w} \in V$

IP2 $\langle c\underline{u}, \underline{v} \rangle = c \langle \underline{u}, \underline{v} \rangle$

IP3 $\langle \underline{u}, \underline{v} \rangle = \langle \underline{v}, \underline{u} \rangle$

IP4 $\langle \underline{v}, \underline{v} \rangle > 0 \quad \forall \underline{v} \in V \setminus \{\underline{0}\}$

A vector space over $\mathbb{R}$ together with a real inner product $\langle \, , \, \rangle : V \times V \to \mathbb{R}$ is a real inner product space.

$\underline{\text{Ex}}$ $\mathbb{R}^n$ with $\langle \underline{x}, \underline{y} \rangle := \underline{x} \cdot \underline{y}$

To see this we check the axioms.

- $\langle \underline{x}+\underline{y}, \underline{z} \rangle = (\underline{x}+\underline{y}) \cdot \underline{z} = \underline{x} \cdot \underline{z} + \underline{y} \cdot \underline{z}$
- $\langle c\underline{x}, \underline{y} \rangle = (c\underline{x}) \cdot \underline{y} = c(\underline{x} \cdot \underline{y})$
- $\langle \underline{x}, \underline{y} \rangle = \underline{x} \cdot \underline{y} = \underline{y} \cdot \underline{x} = \langle \underline{y}, \underline{x} \rangle$
- $\langle \underline{x}, \underline{x} \rangle = \underline{x} \cdot \underline{x} = x_1^2 + \dots + x_n^2 > 0$ if $x \neq 0$

$\underline{\text{Ex}}$ $V = C[a,b]$ over $\mathbb{R}$ with
$\langle f, g \rangle := \int_a^b f(x) g(x)\, dx$

To see that this is an inner product we check the axioms.

- $\langle f+g, h \rangle = \int_a^b (f(x)+g(x)) h(x)\, dx$
  $= \int_a^b f(x)h(x)\, dx + \int_a^b g(x)h(x)\, dx$
  $= \langle f, h \rangle + \langle g, h \rangle$
- $\langle cf, g \rangle = \int_a^b cf(x)g(x)\, dx = c\int_a^b f(x)g(x)\, dx$
  $= c\langle f, g \rangle$
- $\langle f, g \rangle = \int_a^b f(x)g(x)\, dx = \int_a^b g(x)f(x)\, dx$
  $= \langle g, f \rangle$
- $\langle f, f \rangle = \int_a^b (f(x))^2\, dx > 0$ $\quad f(x) \neq 0\ (x)$

<u>Prop</u> Let V be an inner product space over $\mathbb{R}$. For $\underline{u}, \underline{v}, \underline{w} \in V$ & $c \in \mathbb{R}$ the following are true:

1) $\langle \underline{u}, \underline{v}+\underline{w} \rangle = \langle \underline{u}, \underline{v} \rangle + \langle \underline{u}, \underline{w} \rangle$

2) $\langle \underline{u}, c\underline{v} \rangle = c\langle u, v \rangle$

3) $\langle \underline{u}, \underline{u} \rangle = 0$ iff $\underline{u} = 0$

4) if $\langle \underline{x}, \underline{u} \rangle = \langle \underline{x}, \underline{v} \rangle \ \forall \underline{x} \in V$ then $\underline{u} = \underline{v}$

<u>Proof</u>

1) $\langle \underline{u}, \underline{v}+\underline{w} \rangle \overset{IP3}{=} \langle \underline{v}+\underline{w}, \underline{u} \rangle \overset{IP1}{=} \langle \underline{v}, \underline{u} \rangle + \langle \underline{w}, \underline{u} \rangle$

$\overset{IP3'}{=} \langle \underline{u}, \underline{v} \rangle + \langle \underline{u}, \underline{w} \rangle \overset{IP1}{=} \langle \underline{u}, \underline{v}+\underline{w} \rangle$

2) $\langle \underline{u}, c\underline{v} \rangle \overset{IP3'}{=} \langle c\underline{v}, \underline{u} \rangle \overset{IP2'}{=} c\langle \underline{v}, \underline{u} \rangle \overset{IP3'}{=} c\langle \underline{u}, \underline{v} \rangle$

3) $\langle \underline{u}, \underline{u} \rangle = 0$ iff $\underline{u} = 0$

if $\langle \underline{u}, \underline{u} \rangle = 0$ then $\underline{u} \notin V \setminus \{0\}$ by IP4

$\langle \underline{0}, \underline{0} \rangle = \langle \underline{0}+\underline{0}, \underline{0} \rangle \overset{IP1}{=} \langle \underline{0}, \underline{0} \rangle + \langle \underline{0}, \underline{0} \rangle$

$\Rightarrow \langle \underline{0}, \underline{0} \rangle = 0$

<u>Def</u> Let V be an inner product space. We define

$\|\underline{v}\| = \sqrt{\langle \underline{v}, \underline{v} \rangle}$. $\|\underline{v}\|$ is called the length or norm of $\underline{v} \in V$.

Ex Find $\|\underline{x}\|$ for $\underline{x} = (1,1,1,1) \in \mathbb{R}^4$

$$\|\underline{x}\| = \sqrt{<\underline{x},\underline{x}>} = \sqrt{4} = 2$$

Ex Find $\|f(x)\|$ for $f(x) = 1+x \in C[0,1]$

$$\|f(x)\| = \sqrt{<f(x), f(x)>}$$
$$= \sqrt{\int_0^1 (f(x))^2\, dx} = \sqrt{\int_0^1 (1+x)^2\, dx}$$
$$= \sqrt{\int_0^1 1+2x+x^2\, dx}$$
$$= \sqrt{\left[x + x^2 + \tfrac{1}{3}x^3\right]_0^1} = \sqrt{1+1+\tfrac{1}{3}} = \sqrt{\tfrac{7}{3}}$$

Prop Let V be an inner product space. For $\underline{u}, \underline{v} \in V$ & $c \in \mathbb{R}$ the following statements hold:

a) $\|c\underline{v}\| = |c|\,\|\underline{v}\|$

b) $\|\underline{v}\| \geq 0$ & $\|\underline{v}\| = 0$ iff $\underline{v} = \underline{0}$

Proof a) $\|c\underline{v}\| = \sqrt{<c\underline{v}, c\underline{v}>}$

Prop 3.1 + IP2'

$$= \sqrt{c^2 <\underline{v},\underline{v}>} = |c| \sqrt{<\underline{v},\underline{v}>} = |c|\,\|\underline{v}\|$$

b) Immediate from prop 3.1(c)

Cauchy-Schwarz inequality (thm)

Let V be an inner product space.
For $\underline{u}, \underline{v} \in V$ the following inequality holds:

$$|\langle \underline{u}, \underline{v} \rangle| \leq \|\underline{u}\| \, \|\underline{v}\|$$

Furthermore, $|\langle \underline{u}, \underline{v} \rangle| = \|\underline{u}\| \, \|\underline{v}\|$ iff $\underline{u}$ is a scalar multiple of $\underline{v}$.

~~Proof~~

in $\mathbb{R}^n$, $\underline{x} . \underline{y} = \|\underline{x}\| \, \|\underline{y}\| \cos\theta$
$\leq$

Ex Show that

$$\left|\left(\int_{-1}^{1} x f(x)\, dx\right)\right|^2 \leq \frac{2}{3}\int_{-1}^{1} (f(x))^2 \, dx$$

for $f(x) \in C[-1,1]$.

In $C[-1,1]$ $\quad \langle g(x), f(x) \rangle = \int_{-1}^{1} f(x) g(x)\, dx$

$|\langle g(x), f(x) \rangle| \leq \|f(x)\| \, \|g(x)\|$ by C.S.I

$\Rightarrow |\langle g(x), f(x) \rangle|^2 \leq \|f(x)\|^2 \|g(x)\|^2 \quad (*)$

For $g(x) = x$, $(*)$ gives

$$\left|\left(\int_{-1}^{1} x f(x)\, dx\right)\right|^2 \leq \left(\int_{-1}^{1} x^2 \, dx\right)\left(\int_{-1}^{1} (f(x))^2 dx\right)$$

$$= \frac{2}{3}\int_{-1}^{1} (f(x))^2 \, dx$$