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Recall vector spaces $V_1 \xrightarrow{L} V_2 \xrightarrow{T} V_3$
bases $\quad B \quad C \quad D$

$$(\ast)\ [T \circ L]^D_B = [T]^D_C [L]^C_B$$

Prop 2.11 proof

$$[I_W]^{C'}_C [L]^C_B [I_V]^B_{B'}$$

$$\overset{(\ast)}{=} \cancel{[I_W \circ L]^{C'}_B}\ [I_W]^{C'}_C [L \circ I_V]^C_{B'}$$

$$\overset{(\ast)}{=} [I_W \circ L \circ I_V]^{C'}_{B'} = [L]^{C'}_{B'}$$

I_V & I_W are identity

Def Let $L: V \to V$ be a linear operator on a vector space V over a field $\mathbb{F}$. A vector $\underline{v} \in V \setminus \{\underline{0}\}$ is called an eigenvector if there exists a λ ~~in~~ $\in \mathbb{F}$ such that $L(\underline{v}) = \lambda \underline{v}$. The scalar λ is called the eigenvalue corresponding to the eigenvector $\underline{v}$.

Proposition Let $L: V \to V$ be a linear operator on a finite dimensional vector space V & let B, B' be bases for V. The following hold:

a) $\det([L]^B_B) = \det([L]^{B'}_{B'})$

b) $\operatorname{trace}([L]^B_B) = \operatorname{trace}([L]^{B'}_{B'})$

c) $\det([L]^B_B - \lambda [I_V]^B_B) = \det([L]^{B'}_{B'} - \lambda [I_V]^{B'}_{B'})$

Proof (a) determinant, trace & characteristic polynomials are invariant under conjugation.

i.e. If $A = PBP^{-1}$

then $\det(A) = \det(B)$ $\left(\times \frac{\det(P)}{\det(P)}\right)$

then $\operatorname{tr}(A) = \operatorname{tr}(B)$

& $\chi_A(t) = \chi_B(t)$

$L: \underset{B}{V} \to \underset{C}{W}$ if L is invertible then

$\left([L]_B^C\right)^{-1} = [L^{-1}]_C^B$ (prop 2.10)

$\det\left([L]_B^B\right) = \det\left([I_V]_B^{B'}[L]_B^B[I_V]_{B'}^B\right)$ (*)

$= \det\left([I_V \circ L \circ I_V]_{B'}^{B'}\right) = \det\left([L]_{B'}^{B'}\right)$

Proof of (b) & (c) are similar.

Ex Let $L: \mathbb{R}[x]_{\leq 2} \to \mathbb{R}[x]_{\leq 2}$ be the linear map defined by

$$p(x) \mapsto p(x) + (1+p)\,p'(x)$$

Find $\operatorname{tr}(L)$, $\det(L)$, the eigenvalues & eigenvectors.

We set $B = \{1, x, x^2\}$ to be an ordered basis.

$$[L]_B^B := \left([L(1)]_B \quad [L(x)]_B \quad [L(x^2)]_B \right)$$

$$L(1) = 1 \Rightarrow [L(1)]_B = (1\ 0\ 0)^T$$

$$L(x) = 2x + 1 \Rightarrow [L(x)]_B = (1\ 2\ 0)^T$$

$$L(x^2) = 3x^2 + 2x \Rightarrow [L(x^2)]_B = (0\ 2\ 3)^T$$

$$\therefore [L]_B^B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

$\Rightarrow tr(L) = 6 = 1 + 2 + 3$

$det(L) = 6 = 1 \times 2 \times 3$ because

$$\chi_L(\lambda) = det \begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{pmatrix}$$

$$= (1-\lambda)(2-\lambda)(3-\lambda)$$

So $\chi_L(\lambda) = 0 \Leftrightarrow \lambda = 1, \lambda = 2$ or $\lambda = 3$

$\underline{e}_1 = (1\ 0\ 0)^T$ is an eigenvector for $\lambda = 1$

Ex $\underline{e}_2 = (1\ 1\ 0)^T$ is an e-v for $\lambda = 2$

$\underline{e}_3 = (1\ 2\ 1)^T$ is an e-v for $\lambda = 3$

<u>test</u>: $\underline{e}_3$ corresponds to $1+2x+x^2$

$$L(1+2x+x^2) = 1+2x+x^2 + (1+x)(2+2x)$$

$$= 3+6x+3x^2 = 3(1+2x+x^2)$$

For $C = \{1+x,\ 1+x^2,\ x^2\}$

$$L(1+x) = 2+2x = 2(1+x)$$

$$L(1+x^2) = 1+2x+3x^2 = 2(1+x) - 1(1+x^2) + 4(x^2)$$

$$L(x^2) = 2x+3x^2 = 2(1+x) - 2(1+x^2) + 5(x^2)$$

$$[L]_C^C = \begin{pmatrix} 2 & 2 & 2 \\ 0 & -1 & -2 \\ 0 & 4 & 5 \end{pmatrix}$$

$\mathrm{tr}(L) = 6$

$\det(L) = 6$

$\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$

$\cancel{\underline{e}_1 = (1\ 0\ 0)^T}$

$\cancel{\underline{e}_2 = (1\ 1\ 0)^T}$

$\cancel{\underline{e}_3 = (1\ 2\ 1)^T}$

by prop 2.12

<u>Ex</u> Find eigenvectors $\underline{e}_1, \underline{e}_2$ & $\underline{e}_3$