24/3

<u>Prop</u> Let $I_V : V \to V$ be the identity map between vector spaces, & let $B = \{\underline{v}_1, \dots, \underline{v}_n\}$ & $B' = \{\underline{v}_1', \dots, \underline{v}_n'\}$ be bases for V. For every $\underline{v} \in V$;

$$[I_V]_B^{B'} [\underline{v}]_B = [\underline{v}]_{B'}$$

<u>Proof</u> $I_V : V \to V$ $\quad \underline{v} \mapsto I_V(\underline{v}) = \underline{v}$

Let B & B' be bases for V then prop 2.8 tells us

$$[I_V]_B^{B'} [\underline{v}]_B = [I_V(\underline{v})]_{B'} = [\underline{v}]_{B'}$$

<u>Def</u> The matrix $[I_V]_B^{B'}$ in prop 2.9 is called the change of basis matrix.

<u>Ex</u> $B = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ & $B' = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ are bases for $\mathbb{R}^2$.

Let $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbb{R}^2$.

Note, $[\underline{v}]_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$[I]_B^{B'} := \left([I(\underline{v}_1)]_{B'} \; [I(\underline{v}_2)]_{B'} \right) = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$

$[I_V]_B^{B'} [\underline{v}]_B = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} = [\underline{v}]_{B'}$

as $[\underline{v}]_{B'} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ because $\underline{v} = -2\underline{u}_1 + 3\underline{u}_2$

(*) Fact: $V_1 \xrightarrow{L} V_2 \xrightarrow{T} V_3$

bases: B C D

$$[T \circ L]^D_B = [T]^D_C [L]^C_B$$

Exercise composition of two linear maps is linear

Proof Tricky matrix multiplication

Prop Let $L: V \to W$ be a linear transformation between ~~..~~ finite vector spaces & let B & C be bases for V & W respectively. The following are true:

a) L is invertible iff $[L]^C_B$ is invertible

b) If L is invertible then $[L^{-1}]^B_C = ([L]^C_B)^{-1}$

Proof Suppose L is invertible $L: V \to W$ (ie $\exists\, L^{-1}: W \to V$ s.t. $L \circ L^{-1} = I_W$ & $L^{-1} \circ L = I_V$)

$L: V \to W$

bases B C

$$\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \quad I_n = [I_V]^B_B = [L^{-1} \circ L]^B_B$$

$$\overset{(*)}{=} [L^{-1}]^B_{C} [L]^{C}_B$$

ie $([L]^C_B)^{-1} = [L^{-1}]^B_C$

Suppose $A = [L]^C_B$ is invertible.

ie. $A^{-1} = (a_{ij})$ exists

Define $T: W \to V$ by $T(\underline{w}_i) = \sum_{k=1}^{n} a_{ki} \underline{v}_i$

(Let $B = \{\underline{v}_1, \ldots, \underline{v}_n\}$ & $C = \{\underline{w}_1, \ldots, \underline{w}_n\}$ be bases on V & W respectively.)

ie. $[T]_C^B = A^{-1}$

$$[T \circ L]_B^B \overset{(*)}{=} [T]_C^B [L]_B^C = A^{-1}A = I_n$$

ie. $T \circ L = I_V$

Similarly, $L \circ T = I_W$

so L is invertible

<u>Prop</u> Let $L: V \to W$ be a linear transformation between finite dimensional vector spaces & let B_1, B_2 be bases for V & C_1, C_2 be bases for W. The following is true:

$$[L]_{B'}^{C'} = [I_W]_C^{C'} [L]_B^C [I_V]_{B'}^B$$

~~Proof~~