2-7-7

2013

Homework: Q8, Q10, Q19, Q29 For Tuesday 25th

Study Session: Wed 26th March
LT2 or LT7
15:00

Ex $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$

& $B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$

$[A]_B = (a \quad b \quad c \quad d)^T$

where $A = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

ie. $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+b & b+d \\ c & a-b \end{pmatrix}$

① $a+b=2$ ② $b+d=1$

③ $c=0$ ④ $a-b=1$

① + ④ $= 2a = 3 \Rightarrow a = 3/2$

$\Rightarrow b = 1/2$

$\Rightarrow d = 1/2$

ie. $[A]_B = \left(\frac{3}{2} \quad \frac{1}{2} \quad 0 \quad \frac{1}{2}\right)^T$.

The order of B is important.

<u>Note</u>: If $B = \{\underline{v}_1, \ldots, \underline{v}_n\}$ is an ordered basis for V & $\underline{u}, \underline{v} \in V$ then $[\underline{u}+\underline{v}]_B = [\underline{u}]_B + [\underline{v}]_B$

$\underline{u} = a_1 \underline{v}_1 + \ldots + a_n \underline{v}_n$ for some $a_i \in \mathbb{F}$

& $\underline{v} = b_1 \underline{v}_1 + \ldots + b_n \underline{v}_n$ for some $b_i \in \mathbb{F}$

because B is a basis for V

$[\underline{u}]_B = (a_1 \ \ldots \ a_n)^T$

$[\underline{v}]_B = (b_1 \ \ldots \ b_n)^T$

$[\underline{u}+\underline{v}]_B = (a_1+b_1 \ \ldots \ a_n+b_n)^T$

If $a \in \mathbb{F}$ then

$$[a\underline{u}]_B = a(a_1 \ \ldots \ a_n)^T = a[\underline{u}]_B$$

<u>Proposition</u> Let V & W be vector spaces with bases $B_V = \{\underline{v}_1, \ldots, \underline{v}_n\}$ & $B_W = \{\underline{w}_1, \ldots, \underline{w}_m\}$. Let $L: V \to W$ be a linear transformation defined by:

$$\underline{v}_i \mapsto L(\underline{v}_i) = c_{i1} w_1 + \ldots + c_{im} \underline{w}_m$$

We define the matrix $[L]_{B_V}^{B_W}$ by

$$[L]_{B_V}^{B_W} = \left([L(\underline{v}_1)]_{B_W} \ \ldots \ [L(\underline{v}_n)]_{B_W} \right)$$

$$= \begin{pmatrix} c_{11} & \cdots & c_{n1} \\ \vdots & \ddots & \vdots \\ c_{1m} & \cdots & c_{nm} \end{pmatrix}$$

For $\underline{v} \in V$ we have the following identity:

$$[L]^{B_W}_{B_V} [\underline{v}]_{B_V} = [L(\underline{v})]_{B_W}$$

<u>Ex</u> $I : \mathbb{R}[x]_{\leq 2} \to \mathbb{R}[x]_{\leq 3}$ & let $B = \{1, x, x^2\}$ be a basis for $\mathbb{R}[x]_{\leq 2}$ & $C = \{1+x, x^2, x^2-1, x^3\}$

$$I(p(x)) = \int_0^x p(t)\, dt$$

<u>Write down</u> $[I]^C_B$

By def, ~~$[I]^C_B = ([I(1)]_C$~~

$$[I]^C_B := \left([I(1)]_C \;\; [I(x)]_C \;\; [I(x^2)]_C \right)$$

$$I(1) = x = a(1+x) + b(x^2) + c(x^2-1) + d(x^3)$$

$$\Rightarrow d = 0 \qquad \begin{aligned} a - c &= 0 \\ b + c &= 0 \\ a &= 1 \end{aligned}$$

ie. $a = 1,\ c = 1,\ b = -1,\ d = 0$

So $[I(1)]_C = (1 \;\; -1 \;\; 1 \;\; 0)^T$

Similarly, $I(x) = \frac{1}{2}x^2$

$= a(1+x) + b(x^2) + c(x^2-1) + d(x^3)$

So, $a = c = d = 0$, $b = \frac{1}{2}$

$[I(x)]_C = (0 \;\; \tfrac{1}{2} \;\; 0 \;\; 0)^T$

Similarly, $I(x^2) = \frac{1}{3}x^3$

$[I(x^2)]_C = (0 \;\; 0 \;\; 0 \;\; \tfrac{1}{3})^T$

So, $$[I]_B^C := \begin{pmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

<u>Proof</u> Want to show $[L(\underline{v})]_{B_W} = [L]_{B_V}^{B_W} [\underline{v}]_{B_V}$

We know $\underline{v} = a_1\underline{v}_1 + \dots + a_n\underline{v}_n$

[linearity] $L(\underline{v}) = \sum_{i=1}^{n} a_i L(\underline{v}_i)$

$[L(\underline{v})]_{B_W} = [\sum_{i=1}^{n} a_i L(\underline{v}_i)]_{B_W}$ (*)

[$[\underline{u}+\underline{v}]_B = [\underline{u}]_B + [\underline{v}]_B$] $= \sum_{i=1}^{n} [a_i L(\underline{v}_i)]_{B_W} = \sum_{i=1}^{n} a_i [L(\underline{v}_i)]_{B_W}$ [$[a\underline{u}]_B = a[\underline{u}]_B$]

But $[\underline{v}]_{B_V} = (a_1 \; \dots \; a_n)^T$

So $$[L]_{B_V}^{B_W}[\underline{v}]_{B_V} = \begin{pmatrix} | & & | \\ [L(\underline{v}_1)]_{B_W} & \dots & [L(\underline{v}_n)]_{B_W} \\ | & & | \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

$= [L(\underline{v})]_{B_W}$ by (*)