277

17/3

Homework Q8, 10, 19, 29

Study Sessions Wed/~~Thurs~~ Afternoon

Ex Let $L: V \to W$ be such that $\dim(\ker(L)) = 4$ & $\dim(\operatorname{Im}(L)) = 5$

Find the dimension of V.

The Rank-Nullity theorem tells us that:

$$\dim(\operatorname{Im}(L)) + \dim(\ker(L)) = \dim(V) \quad (*)$$

when L is linear.

In our case, $\dim(\operatorname{Im}(L)) = 5$ & $\dim(\ker(L)) = 4$.

So, $\dim(V) = 5 + 4 = 9$ from $(*)$

Prop 2.6 Proof

Let $L: V \to W$ be a linear transformation & suppose $\ker(L) = \{\underline{0}\}$ & suppose $L(\underline{u}) = L(\underline{v})$.

$L(\underline{u}) = L(\underline{v}) \iff L(\underline{u}) - L(\underline{v}) = \underline{0}$

$\iff L(\underline{u} - \underline{v}) = \underline{0} \Rightarrow \underline{u} - \underline{v} = \underline{0}$ (because $\ker(L) = \{\underline{0}\}$) $\Rightarrow \underline{u} = \underline{v}$

ie $\ker(L)=\{\underline{0}\} \Rightarrow L$ is injective.

Now suppose that L is injective (ie $L(\underline{u})=L(\underline{v}) \Rightarrow \underline{u}=\underline{v}$).

If $\underline{u}\in\ker(L)$ then $L(\underline{u})=\underline{0}=L(\underline{0})$

$\Rightarrow \underline{u}=\underline{0}$

by injectivity

ie. $\ker(L)=\{\underline{0}\}$.

Hence L injective $\Leftrightarrow \ker(L)=\{\underline{0}\}$

Suppose $L: V\to V$ is a linear map & $\dim(V)=n$.

The Rank-nullity Theorem tells us that

$$\dim(\ker(L))+\dim(\mathrm{Im}(L))=\dim(V) \quad (*)$$

L injective $\Rightarrow \dim(\ker(L))=0$ $\quad \dim(\{\underline{0}\})=0$

$(*) \Rightarrow \dim(\mathrm{Im}(L))=n$

$\mathrm{im}(L)\subseteq V,\ \dim(U)=\dim(V) \Rightarrow U=V$

$\Rightarrow \mathrm{Im}(L)=V$

Hence, injective implies surjective.

If L is surjective, then $\mathrm{Im}(L)=V$ & $\dim(\mathrm{Im}(L))=\dim(V)$. So $(*) \Rightarrow$ $\dim(\ker(L))=0 \Rightarrow \ker(L)=\{\underline{0}\}$ $\Rightarrow L$ is injective.

Proposition If $L : \mathbb{F}^m \to \mathbb{F}^n$ is a linear map then there exists a matrix $A \in M_{n \times m}(\mathbb{R})$ s.t. $L = \varphi_A$, where φ_A is the linear map defined by matrix multiplication by A.

Proof

Recall If $B = \{\underline{v}_i\}$ is a basis for V & $\{\underline{w}_i\} \subseteq W$, then there is a unique linear map T with $T(\underline{v}_i) = \underline{w}_i$

Let $\{\underline{e}_1, \dots, \underline{e}_m\}$ be the standard basis for $\mathbb{F}^m$.

stand. Bas. for $\mathbb{F}^n$.

~~Consider A~~ then $L(\underline{e}_i) = c_{i1}\underline{b}_1 + \dots + c_{in}\underline{b}_n$

$$= \begin{pmatrix} c_{i1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c_{i2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ c_{in} \end{pmatrix} = \begin{pmatrix} c_{i1} \\ \vdots \\ c_{in} \end{pmatrix}$$

Consider $A := \begin{pmatrix} | & | & & | \\ L(\underline{e}_1) & L(\underline{e}_2) & \dots & L(\underline{e}_m) \\ | & | & & | \end{pmatrix}$

But $\varphi_A : x \mapsto Ax$

$L(\underline{e}_i)$ * is the ith column of A.
But $A\underline{e}_i = L(\underline{e}_i)$. Hence,

$\varphi_A(\underline{e}_i) = L(\underline{e}_i) \quad \forall i$

Hence $L = \ell_A$ as they agree on $\{\underline{e}_i\}$

<u>Definition</u> Let $B = \{\underline{b}_1, \dots, \underline{b}_n\}$ be an ordered basis for a vector space V over a field $\mathbb{F}$ & let $c_1, \dots, c_n \in \mathbb{F}$ be such that $\underline{v} = c_1\underline{b}_1 + \dots + c_n\underline{b}_n$. We define

$$[\underline{v}]_B := \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

ORDER IS IMPORTANT

We call $[\underline{v}]_B$ the coordinate vector of $\underline{v}$ relative to the basis B.

<u>Ex</u> Let $B = \{1, 1+x, 1+x+x^2, x^3\}$

& $S = \{1, x, x^2, x^3\}$ be ordered bases for $\mathbb{R}[x]_{\leq 3}$ (& $C = \{1+x+x^2, x^3, 1, 1+x\}$)

For $p(x) = x + x^3$ we have

$$[p(x)]_S = (0 \quad 1 \quad 0 \quad 1)^T$$

$$[p(x)]_B = (-1 \quad 1 \quad 0 \quad 1)^T$$

because $p(x) = -1(1) + 1(1+x)$

$$+ 0(1+x+x^2) + 1(x^3)$$

$$[p(x)]_C = (0 \quad 1 \quad -1 \quad 1)^T \neq [p(x)]_B$$