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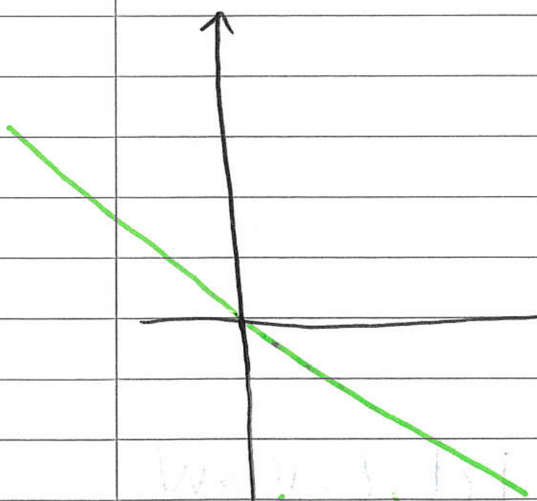
Definition Let $L: V \to W$ be a linear transformation between finite dimentional vector spaces. We call $\dim(\mathrm{im}(L))$ the rank of L & $\dim(\ker(L))$ the nullity of L.

Ex $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

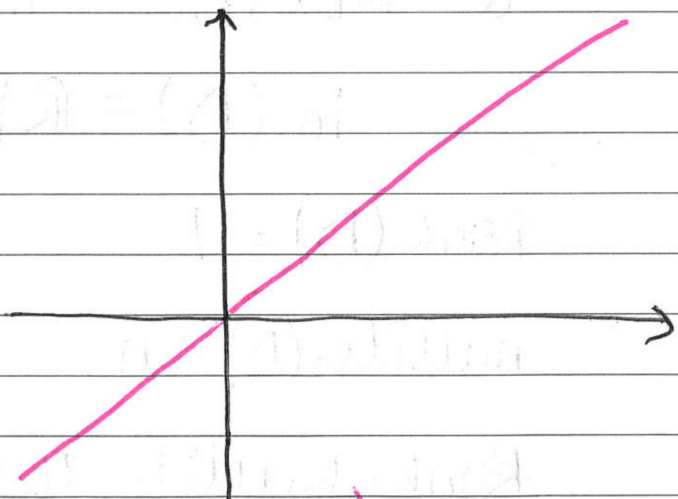
$\varphi_A : \mathbb{R}^2 \to \mathbb{R}^2 \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \end{pmatrix}$

$\ker(\varphi_A) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid x = -y \right\}$

$\mathrm{Im}(\varphi_A) = \left\{ \begin{pmatrix} a \\ a \end{pmatrix} \in \mathbb{R}^2 \mid a \in \mathbb{R} \right\}$



$\ker(\varphi_A)$



$\mathrm{Im}(\varphi_A)$

rank $= \dim(\mathrm{Im}(\varphi_A)) = 1$

nullity $= \dim(\ker(\varphi_A)) = 1$

$\underline{Ex}$ $D : \mathbb{R}[x]_{\leq n} \to \mathbb{R}[x]_{\leq n-1}$

$$p(x) \mapsto p'(x)$$

~~Kernel~~ $\ker(D) = \{a \mid a \in \mathbb{R}\} = \mathbb{R}$

$\mathrm{Im}(D) \subseteq \mathbb{R}[x]_{\leq n-1}$

If $p(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} \in \mathbb{R}[x]_{\leq n-1}$

but $D(a_0 x + \frac{1}{2} a_1 x^2 + \ldots + \frac{1}{n} a_{n-1} x^n)$

$$= p(x)$$

ie. $\mathbb{R}[x]_{\leq n-1} \subseteq \mathrm{Im}(D)$

$$\mathrm{Im}(D) = \mathbb{R}[x]_{\leq n-1}$$

$\mathrm{rank}(D) = 1$

$\mathrm{nullity}(D) = n$

<u>Rank-Nullity Theorem</u> Let $L : V \to W$ be a linear transformation between vector spaces. If V is finite dimensional then

$$\dim(\ker(L)) + \dim(\mathrm{Im}(L)) = \dim(V)$$

<u>Proof</u> Let $\dim(V) = n$. &

~~$\{\underline{v}_1, \ldots, \underline{v}_n\}$ be a basis for V.~~

Let $\dim(\ker(L)) = k$ & let
$B = \{\underline{v}_1, \dots, \underline{v}_k\}$ be a basis for $\ker(L)$

Recall S.E.L Every linearly independent set in a finite dimensional vector space can be extended to a basis.

We can extend B to a basis $\{\underline{v}_1, \dots, \underline{v}_k, \underline{v}_{k+1}, \dots, \underline{v}_n\}$ for V.

Claim: $\{L(\underline{v}_{k+1}), \dots, L(\underline{v}_n)\}$ is a basis for $\mathrm{Im}(L)$. To show this, we need to show that $\{L(\underline{v}_{k+1}), \dots, L(\underline{v}_n)\}$ spans $\mathrm{Im}(L)$ & is linearly independent.

We will first show that
$C = \{L(\underline{v}_1), \dots, L(\underline{v}_n)\}$ spans $\mathrm{Im}(L)$

Let $\underline{w} \in \mathrm{Im}(L) \Rightarrow \underline{w} = L(\underline{v})$ where $\underline{v} = c_1\underline{v}_1 + \dots + c_n\underline{v}_n \in V$.

$\underline{w} = L(\underline{v}) = L(c_1\underline{v}_1 + \dots + c_n\underline{v}_n)$

by linearity

$= c_1 L(\underline{v}_1) + \dots + c_n L(\underline{v}_n) \in sp(L(\underline{v}))$

ie. C spans $\mathrm{Im}(L)$

But $L(\underline{v}_1) = \dots = L(\underline{v}_k) = 0$ because $\underline{v}_1, \dots, \underline{v}_k \in \ker(L)$ hence ~~sp~~

$sp\{L(\underline{v}_{k+1}), \dots, L(\underline{v}_n)\} = sp\{L(\underline{v}_1), \dots, L(\underline{v}_n)\} = \mathrm{Im}(L)$

~~Suppose $c_1 L(\underline{v}_1) + ...$~~

Suppose $c_{k+1} L(\underline{v}_{k+1}) + \dots + c_n L(\underline{v}_n) = \underline{0}$

linearity

$\Rightarrow L(\underbrace{c_{k+1}\underline{v}_{k+1} + \dots + c_n \underline{v}_n}_{\underline{u}}) = \underline{0}$

$\Rightarrow \underline{u} \in \ker(L)$

$\Rightarrow \underline{u} = c_{k+1}\underline{v}_{k+1} + \dots + c_n \underline{v}_n$

$= d_1 \underline{v}_1 + \dots + d_k \underline{v}_k$ for some $d_i \in \mathbb{F}$

$\Rightarrow c_i d_j = 0$ because $\{\underline{v}_1, \dots, \underline{v}_n\}$ is linearly independent because $\{\underline{v}_1, \dots, \underline{v}_n\}$ is a basis for V $\therefore$ Lin ind by definition

~~$\Rightarrow \{L(\underline{v}_1)$~~

$\Rightarrow \{L(\underline{v}_{k+1}), \dots, L(\underline{v}_n)\} = D$ is linearly independent

$\Rightarrow D$ is a basis for $\mathrm{Im}(L)$

$\dim(\ker(L)) + \dim(\mathrm{Im}(L))$

$= k + (n-k) = n = \dim(V)$

<u>Proposition</u> Let $L: V \to W$ be a linear transformation between vector spaces. The map L is one-to-one if & only if $\ker(L) = \{0\}$. Moreover, if $V = W$ & V is finite dimensional then L is one-to-one iff L is onto.

Ex $L : \mathbb{R}[x]_{\leq n} \to \mathbb{R}[x]$

$$p(x) \mapsto 2p'(x) + \int_0^{x^2} p(x)\,dx$$

Exercise: L is linear

Find $\dim(\mathrm{Im}(L))$

Rank-nullity: $\Rightarrow \dim(\mathrm{Im}(L)) + \dim(\ker(L))$

$$= \dim(\mathbb{R}[x]_{\leq n}) \quad (*)$$

$$\dim(\mathrm{Im}(L)) = \dim(\mathbb{R}[x]_{\leq n}) - \dim(\ker(L))$$

If $p(x)$ has degree n then

$\deg(L(p(x))) = 2n+2$ for $p(x) \neq 0$

$\Rightarrow p(x) = 0$ is the only polynomial that maps to zero.

$\Rightarrow \dim(\ker(L)) = 0$

& hence $\dim(\mathrm{Im}(L)) = n+1$

from $(*)$

~~Proposition Let $L: V \to W$ be a linear transformation between vector spaces. The ima~~