MAS277 Lecture 1.

Office hours - Wednesday Morning.

$\mathbb{F}$ = field $\quad \mathbb{F}, +, \cdot \quad +: \mathbb{F} \times \mathbb{F} \to \mathbb{F}$

A field is a set together with the properties

F1: $\mathbb{F}$ is an abelian group with respect to the binary operation '+'

Binary operation means closure must hold.

other group axioms
- associativity
- identity
- Inverses.
- commutativity - since abelian.

Notation $+(a,b)$ $\quad e_+ \quad a^{-1}$ w.r.t $+ \quad a+(-b)$

write → "$a+b$" $\quad 0$ (identity) $\quad -a \quad a-b$

remove 0

F2: $\mathbb{F}\setminus\{e_+\}$ is an Abelian group with respect to the binary operation "$\cdot$"

$\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$

write → $\cdot(a,b) \quad e_\cdot \quad a^{-1}$ w.r.t $\cdot \quad a^{-1}b$

"$a \cdot b$" "ab" $\quad 1 \quad a^{-1}, \frac{1}{a} \quad \frac{b}{a}$

F3: Multiplication is distributive w.r.t addition.
ie for every $a, b, c \in \mathbb{F}$ we have

$$a \cdot (b+c) = a \cdot b + a \cdot c$$

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Notation	$+(a,b)$	e_+	a^{-1} w.r.t $+$	$a+(-b)$
write $\to$	"$a+b$"	0 (identity)	$-a$	$a-b$

F2: $F \setminus \{e_+\}$ is an Abelian group with respect to the binary operation "$\cdot$" (remove 0)

$\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$

write $\to$	$\cdot(a,b)$	$e_\cdot$	a^{-1} w.r.t $\cdot$	$a^{-1}b$
	"$a \cdot b$" "ab"	1	$a^{-1}, \frac{1}{a}$	$\frac{b}{a}$

F3: Multiplication is distributive w.r.t addition.
ie for every $a, b, c \in \mathbb{F}$ we have

$$a \cdot (b+c) = a \cdot b + a \cdot c$$

Examples :

1. $\mathbb{R}$ is a field
   sol : • $(\mathbb{R}, +)$ is an Abelian group.

   • $(\mathbb{R} \setminus \{0\}, \cdot)$ is an Abelian group.

   • $a(b+c) = ab + ac \quad \forall a, b, c \in \mathbb{R}$

   Hence $\mathbb{R}$ is a field with the usual notions of addition & subtraction.

2. $\mathbb{Z}$ taken with addition & multiplication is not a field
   proof :
   $(\mathbb{Z}, \cdot)$ is not an Abelian group as 2 does not have an inverse in $\mathbb{Z}$

   multiplication.

3. $\mathbb{Q}$ is a field $\qquad \mathbb{Z}/p\mathbb{Z}$ p is prime - Integers ~~mod P~~
   $\mathbb{C}$ is a field.

4. $\mathbb{R} \setminus \mathbb{Q}$ (Irrational numbers) is not a field as it is not closed w.r.t multiplication.

   eg $\sqrt{2} \cdot \sqrt{2} = 2 \in \mathbb{R} \setminus \mathbb{Q}$

Proposition 1.1 :

For a, b, c elements of a field $\mathbb{F}$ we have :

a) $a+b = c+b \Rightarrow a = c$
b) $a.b = c.b \Rightarrow a = c$ when $b \neq 0$
c) $a.0 = 0$
d) $(-1).a = -a$
e) $(-a).b = a.(-b) = -(a.b)$

Proof of (a) : $a+b = c+b \Rightarrow$

$$(a+b)+(-b) = (c+b) - b$$

Shorthand notation for $+(-b)$

$\Downarrow$ by associativity of F1

$$a+(b-b) = c+(b-b)$$

$$\underset{\text{by Inverses F1}}{\Rightarrow} \quad a+0 = c+0 \quad \underset{\text{Identity F1}}{\Rightarrow} \quad a = c$$

Proof of (b) : Similar to (a)

Proof of (c) : $a.0 = \underset{\text{identity F1}}{a.0+0} \underset{\text{Identity F1}}{=} a(0+0)$

$$\underset{\text{distributive F3}}{=} a.0 + a.0 \underset{\text{Implies by (a)}}{\Rightarrow} 0 = a.0$$