Solutions for MAS277 Problems

Solutions for Chapter 3 problems: Inner product spaces

1. No. The only part that fails, however, is the condition that $\langle f, f \rangle = 0$ should mean that $f = 0$. If f is a non-zero function that vanishes at $x = 0$ (e.g., $f(x) = x$), then $\langle f, f \rangle = 0$ but $f \neq 0$.

2. We see that

$$\|x + y\|^2 = \langle x + y, x + y \rangle = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle,$$

and

$$\|x - y\|^2 = \langle x - y, x - y \rangle = \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle,$$

using our rules for inner products. Then

$$\|x + y\|^2 + \|x - y\|^2 = 2\langle x, x \rangle + 2\langle y, y \rangle = 2\|x\|^2 + 2\|y\|^2,$$

as required.

3. (a) We have

$$\begin{array}{lll} \langle C_1, C_1 \rangle = 6, & \langle C_1, C_2 \rangle = 10, & \langle C_1, C_3 \rangle = 6, \\ \langle C_2, C_1 \rangle = 10, & \langle C_2, C_2 \rangle = 37, & \langle C_2, C_3 \rangle = 0, \\ \langle C_3, C_1 \rangle = 6, & \langle C_3, C_2 \rangle = 0, & \langle C_3, C_3 \rangle = 28. \end{array}$$

(b) As $\langle \ , \ \rangle$ is a real inner product, $\langle A, B \rangle = \langle B, A \rangle$. But

$$\langle B, A \rangle = \mathrm{trace}(BA^T) = \mathrm{trace}(BA) = \mathrm{trace}(AB) = -\mathrm{trace}(AB^T) = -\langle A, B \rangle,$$

so $\langle A, B \rangle = -\langle A, B \rangle$, and so $\langle A, B \rangle = 0$. (Alternatively, one could expand out the products, but this is simpler.)

4. (a) We have

$$\langle x{+}1, x^2{+}x \rangle = \int_{-1}^{1} (x{+}1)(x^2{+}x)\, dx = \int_{-1}^{1} x^3{+}2x^2{+}x\, dx = \left[\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} \right]_{-1}^{1} = \frac{4}{3}.$$

(b) We have

$$\langle x^i, x^j \rangle = \int_{-1}^{1} x^{i+j}\, dx = \left[\frac{x^{i+j+1}}{i+j+1} \right]_{-1}^{1} = 0$$

since $i + j + 1$ is even, meaning that $(-1)^{i+j+1} = 1^{i+j+1}$.

(c) We have

$$4f(-1) - 8f(0) + 4f(1) = 4(a - b + c) - 8c + 4(a + b + c) = 8a.$$

Also,

$$
\begin{aligned}
\langle f, u\rangle &= \int_{-1}^{1} (px^2+q)(ax^2+bx+c)\,dx \\
&= \int_{-1}^{1} apx^4 + bpx^3 + (pc+qa)x^2 + bqx + cq\,dx \\
&= \left[\frac{apx^5}{5} + \frac{bpx^4}{4} + \frac{(pc+qa)x^3}{3} + \frac{bqx^2}{2} + cqx\right]_{-1}^{1} \\
&= \frac{2ap}{5} + \frac{2(pc+qa)}{3} + 2cq \\
&= a\left(\frac{2p}{5} + \frac{2q}{3}\right) + c\left(\frac{2p}{3} + 2q\right).
\end{aligned}
$$

We require that this equal $8a$, so we need

$$
\begin{aligned}
\frac{2p}{5} + \frac{2q}{3} &= 8 \\
\frac{2p}{3} + 2q &= 0,
\end{aligned}
$$

and solving this as usual gives $p = 45$, $q = -15$.

5. We have

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle = \|x\|^2 + 2\langle x,y\rangle + \|y\|^2,$$

so the given formula holds if and only if $\langle x, y\rangle = 0$, as required.

6. (a) Notice that, for $f, g \in C^\infty(\mathbb{R})$, $\langle f, g\rangle$ is a function. So $\langle\ ,\ \rangle$ does not necessarily take real number values, and is not therefore an inner product. But restricting to V, things are better:

$$\langle f, g\rangle' = (fg + f'g')' = f'g + fg' + f''g' + f'g'' = f'(g+g'') + g'(f+f'') = 0,$$

as both f and g lie in V, so that $f + f'' = 0$ and $g + g'' = 0$. So $\langle f, g\rangle$ is a constant function, that is, a real number.

(b) If $f + f'' = 0$, then $(f + f'')' = 0$, so $f' + f''' = 0$, i.e., $f' + (f')'' = 0$, and so $f' \in V$.

(c) Clearly $\langle\ ,\ \rangle$ satisfies conditions IP1–IP3 of the definition of inner product, and is also valued in $\mathbb{R}$ by part (a). Also, a typical element in V is $f = a\sin t + b\cos t$. Then

$$\langle f, f\rangle = f^2 + f'^2 = (a\sin t + b\cos t)^2 + (a\cos t - b\sin t)^2 = a^2 + b^2,$$

which is clearly non-negative, and is only zero when $a = b = 0$, i.e., when $f = 0$ as required.

(d) As $D(\sin) = \cos$ and $D(\cos) = -\sin$, the matrix representation of D relative to the basis $\{\sin, \cos\}$ is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

7. We use the Cauchy-Schwarz inequality on the space $C[0,1]$ with its usual inner product. Put $g(x) = 1 + x$. Then

$$\langle g, g \rangle = \int_0^1 g(x)^2\,dx = \int_0^1 (1+x)^2\,dx = \int_0^1 x^2 + 2x + 1\,dx = \left[\frac{x^3}{3} + x^2 + x\right]_0^1 = \frac{7}{3}.$$

The Cauchy-Schwarz inequality tells us that

$$\left|\int_0^1 f(x)(1+x)\,dx\right| \le \langle 1+x, 1+x \rangle \langle f, f \rangle,$$

which is exactly the statement required. The Cauchy-Schwarz inequality is an equality if $f(x)$ is a scalar multiple of $1+x$, so $f(x) = 1+x$ is one function where we have equality.

8. We use the Cauchy-Schwarz inequality on the space $C[-1,1]$ with its usual inner product. Put $g(x) = \sqrt{1-x^2}$. Then

$$\langle g, g \rangle = \int_{-1}^1 g(x)^2\,dx = \int_{-1}^1 1 - x^2\,dx = \left[x - \frac{x^3}{3}\right]_{-1}^1 = \frac{4}{3}.$$

The Cauchy-Schwarz inequality tells us that

$$\left|\int_{-1}^1 f(x)\sqrt{1-x^2}\,dx\right| \le \langle \sqrt{1-x^2}, \sqrt{1-x^2} \rangle \langle f, f \rangle,$$

which is exactly the statement required. The Cauchy-Schwarz inequality is an equality if $f(x)$ is a scalar multiple of $\sqrt{1-x^2}$, so $f(x) = \sqrt{1-x^2}$ is one function where we have equality.

9. We use the Cauchy-Schwarz inequality on the space $C[0,1]$ with its usual inner product. Given a function $f(x)$, put $g(x) = f(x)^2$. Then

$$\langle g, g \rangle = \int_0^1 g(x)^2\,dx = \int_0^1 f(x)^4\,dx.$$

The Cauchy-Schwarz inequality tells us that

$$\left|\int_0^1 f(x)g(x)\,dx\right| \le \langle g, g \rangle \langle f, f \rangle,$$

which is exactly the statement required when we substitute $g(x) = f(x)^2$. The Cauchy-Schwarz inequality is an equality if $f(x)$ is a scalar multiple of $f(x)^2$, so we need $f(x) = \lambda f(x)^2$, i.e., $f(x) = 1/\lambda$. Thus this is an equality precisely for constant functions. (One ought to be a little more careful about the possibility that $f(x) = 0$ at some points, but as f is continuous, one sees that either $f(x) = 0$ for all x, or $f(x) = 1/\lambda$ for all x.)

10. $f(x) = ax^2 + bx + c$ is orthogonal to 1 and x if and only if $\langle f(x), 1\rangle = \langle f(x), x\rangle$. Using the definition of the inner product we require

$$\langle f(x), 1\rangle = \int_0^1 (ax^2 + bx + c)\mathrm{dx} = \frac{a}{3} + \frac{b}{2} + c = 0$$

and

$$\langle f(x), 1\rangle = \int_0^1 (ax^3 + bx^2 + cx)\mathrm{dx} = \frac{a}{4} + \frac{b}{3} + \frac{c}{2} = 0.$$

Setting $a = 1$ in the above equations we find $b = -1$ and $c = \frac{1}{6}$. So the polynomial $f(x) = x^2 - x + \frac{1}{6}$ is orthogonal to both 1 and x.

11. First, $\phi(a + b\cos t + c\sin t) = b\sin t + c\cos t$, so

$$\begin{aligned} \langle \phi(a + b\cos t + c\sin t), \alpha + \beta\cos t + \gamma\sin t\rangle &= \langle b\sin t + c\cos t, \alpha + \beta\cos t + \gamma\sin t\rangle \\ &= \pi(\beta c + \gamma b). \end{aligned}$$

If ϕ^* is the adjoint of ϕ, we need

$$\langle a + b\cos t + c\sin t, \phi^*(\alpha + \beta\cos t + \gamma\sin t)\rangle = \pi(\beta c + \gamma b)$$

and so $\phi^*(\alpha + \beta\cos t + \gamma\sin t) = \beta\sin t + \gamma\cos t$, showing that $\phi^* = \phi$.

12. If $\mathbf{v} = (x, y, z)^T$, we have

$$\langle \phi(\mathbf{v}), A\rangle = \left\langle \begin{pmatrix} x & y \\ y & z \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\rangle = ax + by + cy + dz.$$

On the other hand, if $\mathbf{w} = (p, q, r)^T$, then

$$\langle \mathbf{v}, \mathbf{w}\rangle = px + qy + rz.$$

These two agree when $p = a$, $q = b + c$ and $r = d$. So $\mathbf{w} = (a, b + c, d)^T$.

13. Let

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix}.$$

We need to find ϕ^* so that $\langle \phi(A), B\rangle = \langle A, \phi^*(B)\rangle$. But

$$\langle \phi(A), B\rangle = a_4 b_2 + a_7 b_3 + a_8 b_6.$$

In order that this is equal to $\langle A, \phi^*(B)\rangle$, we need

$$\phi^* \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ b_2 & 0 & 0 \\ b_3 & b_6 & 0 \end{pmatrix}.$$

14. Explicitly, $\phi\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} a+b+c+d & a+b+c+d \\ a+b+c+d & a+b+c+d\end{pmatrix}$. Let $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}$. Then

$$\langle \phi(A), B\rangle = (a+b+c+d)(\alpha+\beta+\gamma+\delta).$$

We need this to equal $\langle A, \phi^*(B)\rangle$, and as this is

$$a(\alpha+\beta+\gamma+\delta)+b(\alpha+\beta+\gamma+\delta)+c(\alpha+\beta+\gamma+\delta)+d(\alpha+\beta+\gamma+\delta),$$

we need $\phi^*(B) = \begin{pmatrix} \alpha+\beta+\gamma+\delta & \alpha+\beta+\gamma+\delta \\ \alpha+\beta+\gamma+\delta & \alpha+\beta+\gamma+\delta\end{pmatrix}$, and so $\phi^* = \phi$, as required.

15. We see $\chi(ax^2+bx+c) = 2a$. If $u = \alpha x^2 + \beta x + \gamma$, then

$$\begin{aligned}
\langle f, u\rangle &= \int_{-1/2}^{1/2} (ax^2+bx+c)(\alpha x^2+\beta x+\gamma)\,dx \\
&= \int_{-1/2}^{1/2} a\alpha x^4 + (a\beta+\alpha b)x^3 + (a\gamma+b\beta+c\alpha)x^2 + (b\gamma+\beta c)x + c\gamma\,dx \\
&= \left[\frac{a\alpha x^5}{5} + \frac{(a\beta+\alpha b)x^4}{4} + \frac{(a\gamma+b\beta+c\alpha)x^3}{3} + \frac{b\gamma+\beta c)x^2}{2} + c\gamma x\right]_{-1/2}^{1/2} \\
&= \frac{a\alpha}{80} + \frac{a\gamma+b\beta+c\alpha}{12} + c\gamma \\
&= a\left(\frac{\alpha}{80}+\frac{\gamma}{12}\right) + b\frac{\beta}{12} + c\left(\frac{\alpha}{12}+\gamma\right).
\end{aligned}$$

We require that this always equal $2a$. So $\beta = 0$ and $\frac{\alpha}{12} = -\gamma$. Then $\frac{\alpha}{80} - \frac{\alpha}{144} = 2$, and this gives $\alpha = 360$, and so $\gamma = -30$. Thus $u = 360x^2 - 30$.

We need $\langle \chi(f), t\rangle = \langle f, \chi^*(t)\rangle$ for $t \in \mathbb{R}$. The left-hand side is

$$\langle \chi(f), t\rangle = \langle f''(0), t\rangle = f''(0).t = \langle f, u\rangle t = \langle f, tu\rangle,$$

so $\chi^*(t) = tu = t(360x^2-30)$.

16. We simply observe that for $i \neq j$,

$$\langle \phi(\mathbf{v}_i), \phi(\mathbf{v}_j)\rangle = \langle \mathbf{v}_i, \phi^*(\phi(\mathbf{v}_j))\rangle = \langle \mathbf{v}_i, \mathbf{v}_j\rangle = 0,$$

the first equality being the definition of the adjoint, the middle by the given property of ϕ, and the final one because $\mathbf{v}_i$ and $\mathbf{v}_j$ are orthogonal. Similarly, the same argument gives $\langle \phi(\mathbf{v}_i), \phi(\mathbf{v}_i)\rangle = \langle \mathbf{v}_i, \mathbf{v}_i\rangle = 1$, so $\{\phi(\mathbf{v}_1), \ldots, \phi(\mathbf{v}_n)\}$ is an orthonormal set.

17. (a) Notice that f'' is just a constant. So

$$\begin{aligned}
\alpha(\alpha(f)) = \alpha((3x^2-1)f'') = f''\alpha(3x^2-1) &= f''(3x^2-1)(3x^2-1)'' \\
&= 6f''(3x^2-1) = 6\alpha(f).
\end{aligned}$$

(b) If $\alpha(f) = \lambda f$, then $\alpha(\alpha(f)) = \alpha(\lambda f) = \lambda.\alpha(f) = \lambda^2 f$. But we also have $\alpha(\alpha(f)) = 6\alpha(f) = 6\lambda f$. As f is nonzero, $\lambda^2 = 6\lambda$.

(c) The eigenvectors with eigenvalue 0 are those polynomials where $f'' = 0$, i.e., $f(x) = ax + b$. The eigenvector with eigenvalue 6 must have $\alpha(f) = 6f$; but $\alpha(f)$ is always a scalar multiple of $3x^2 - 1$, and indeed this is an eigenvector with eigenvalue 6. Thus we begin with the sequence $1, x, 3x^2 - 1$. It is easy to see that 1 and x are already orthogonal. And $3x^2 - 1$ is orthogonal to both 1 and x, since two eigenvectors of a self-adjoint operator with different eigenvalues are necessarily orthogonal.

Solutions for Chapter 4 problems: Gram-Schmidt and Fourier theory.

1. Integrating by parts, we have

$$\begin{aligned}\langle t, \sin kt\rangle &= \int_{-\pi}^{\pi} t \sin kt \, dt = \left[-t\frac{\cos kt}{k}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos kt}{k}\, dt \\ &= \left[-\pi\frac{\cos k\pi}{k} + (-\pi)\frac{\cos(-k\pi)}{k}\right] = -\frac{2\pi}{k}\cos k\pi = \frac{(-1)^{k+1}2\pi}{k}.\end{aligned}$$

Notice that computing these inner products involves carrying out exactly the same integrals that you use when computing Fourier series.

2. (a) We use the angle formula:

$$\cos(A+B)t + \cos(A-B)t = 2\cos At \cos Bt$$

to see that $\cos t \cos 4t = (\cos 5t + \cos 3t)/2$. Then

$$\langle \cos 3t, \cos t \cos 4t\rangle = \frac{\langle \cos 3t, \cos 5t\rangle}{2} + \frac{\langle \cos 3t, \cos 3t\rangle}{2} = 0 + \frac{\pi}{2}.$$

On the other hand,

$$\langle \cos 3t, \cos 3t\rangle = \pi,$$

and

$$\begin{aligned}\langle \cos t \cos 4t, \cos t \cos 4t\rangle &= \frac{\langle \cos 5t + \cos 3t, \cos 5t + \cos 3t\rangle}{4} \\ &= \frac{\langle \cos 5t, \cos 5t\rangle}{4} + \frac{\langle \cos 3t, \cos 3t\rangle}{4} = \frac{\pi}{2}.\end{aligned}$$

Combining these, we see that if θ is the angle between the functions, then

$$\cos\theta = \frac{\pi/2}{\sqrt{\pi}\sqrt{\pi/2}} = \frac{1}{\sqrt{2}}.$$

(b) We use the angle formula:

$$\sin(A+B)t + \sin(A-B)t = 2\sin At \cos Bt$$

to see that $\cos 2t \sin t = (\sin 3t - \sin t)/2$ and $\cos 5t \sin t = (\sin 6t - \sin 4t)/2$. Then as $\langle \sin mt, \sin nt\rangle = 0$ if $m \neq n$, we see that $\langle \cos 2t \sin t, \cos 5t \sin t\rangle = 0$, as required.

3. We use the Cauchy-Schwarz inequality. Put $g(x) = \sin x$. Then

$$\langle g, g\rangle = \int_{-\pi}^{\pi} \sin^2 x \, dx = \pi.$$

The Cauchy-Schwarz inequality tells us that

$$\left|\int_{-\pi}^{\pi} \sin x\, f(x)\, dx\right| \leq \langle \sin x, \sin x\rangle\langle f, f\rangle,$$

which is exactly the statement required. The Cauchy-Schwarz inequality is an equality if $f(x)$ is a scalar multiple of $\sin x$, so $f(x) = \sin x$ is one function where we have equality.

4. We put $f_1 = 1$, $f_2 = x$, $f_3 = x^2$. Put $g_1 = f_1 = 1$, and $g_2 = f_2 - \lambda g_1 = x - \lambda$. We wish to choose λ so that g_2 is orthogonal to f_1: we know that $\lambda = \frac{\langle f_2, g_1\rangle}{\langle g_1, g_1\rangle}$, but it is almost easier in this case just to work it out directly:

$$\langle f_1, g_2\rangle = \int_0^1 (x-\lambda)\,dx = \left[\frac{x^2}{2} - \lambda x\right]_0^1 = \frac{1}{2} - \lambda,$$

and so we must choose $\lambda = 1/2$. Thus $g_2 = x - \frac{1}{2}$.

Next, we put $g_3 = f_3 - \lambda g_1 - \mu g_2 = x^2 - \lambda - \mu(x - \frac{1}{2})$, and we want to choose λ and μ so that g_3 is orthogonal to f_1 and to f_2. We can work out λ and μ using the formulae, or just compute the inner products. Let's do that.

$$\langle f_1, g_3\rangle = \int_0^1 x^2 - \lambda - \mu(x - \frac{1}{2})\,dx = \left[\frac{x^3}{3} - \lambda x - \mu\frac{x^2}{2} + \mu\frac{x}{2}\right]_0^1 = \frac{1}{3} - \lambda.$$

So $\lambda = 1/3$. Next,

$$\langle f_2, g_3\rangle = \int_0^1 x^3 - \lambda x - \mu(x - \frac{1}{2})x\,dx = \left[\frac{x^4}{4} - \lambda\frac{x^2}{2} - \mu\frac{x^3}{3} + \mu\frac{x^2}{4}\right]_0^1 = \frac{1}{4} - \frac{\lambda}{2} - \frac{\mu}{12},$$

and so $\frac{1}{4} - \frac{1/3}{2} - \frac{\mu}{12} = 0$, giving $\mu = 1$. Then $g_3 = x^2 - \frac{1}{3} - (x - \frac{1}{2}) = x^2 - x + \frac{1}{6}$.

5. We can take $g_1 = f_1 = \sin t$. Notice that $f_2 = \frac{1}{2}\sin 2t$ is already orthogonal to g_1 as $\langle \sin mt, \sin nt\rangle = 0$ if $m \neq n$. So we can put $g_2 = f_2$, and g_2 is already orthogonal to g_1.

Now notice that $f_3(t) = (\sin 3t - \sin t)/2$ is already orthogonal to $g_2(t)$, again using the remark about $\langle \sin mt, \sin nt\rangle$ above. To make it orthogonal to g_1, put $g_3 = f_3 - \lambda g_1 = \sin t(\cos 2t - \lambda)$, and try to choose λ so that $\langle g_3, g_1\rangle = 0$. But

$$\begin{aligned}\langle g_3, g_1\rangle &= \int_{-\pi}^{\pi} \sin t.\sin t(\cos 2t - \lambda)\,dt \\ &= \int_{-\pi}^{\pi} (1 - \cos 2t)(\cos 2t - \lambda)/2\,dt \\ &= -\pi\lambda - \pi/2,\end{aligned}$$

so $\lambda = -1/2$. Thus $g_3 = \sin t(\cos 2t + 1/2)$.

6. Clearly a basis for V is $B_1 = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, $B_2 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and $B_3 = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$, and these three matrices are clearly orthogonal to each other, so form an orthogonal basis for V. For $A \in V$,

$$\begin{aligned}\pi(A) &= \frac{\langle A, B_1\rangle}{\langle B_1, B_1\rangle}B_1 + \frac{\langle A, B_2\rangle}{\langle B_2, B_2\rangle}B_2 + \frac{\langle A, B_3\rangle}{\langle B_3, B_3\rangle}B_3 \\ &= \frac{a}{1}B_1 + \frac{b+c}{2}B_2 + \frac{d}{1}B_3 \\ &= \begin{pmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d\end{pmatrix} \\ &= (A + A^T)/2\end{aligned}$$

An alternative approach is to identify $V^\perp$ and to write A in its unique form $C+D$, where $C \in V$ and $D \in V^\perp$. Then $\pi(A) = C$. A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is orthogonal to B_1 if $a = 0$; is orthogonal to B_2 if $b+c = 0$, and orthogonal to B_3 if $d = 0$. Then $V^\perp = \left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \;\middle|\; b \in \mathbb{R} \right\}$. Given any matrix A, we want to write it as $C+D$, where $C \in V$ and $D \in V^\perp$. The unique way to do this is with

$$C = \begin{pmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{pmatrix} \text{ and } D = \begin{pmatrix} 0 & \frac{b-c}{2} \\ \frac{c-b}{2} & 0 \end{pmatrix}.$$

Then

$$\pi(A) = C = \begin{pmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{pmatrix} = (A + A^T)/2,$$

as required.

7. Let's take $B_1 = A_1$. To find a matrix $B_2 = A_2 - \lambda B_1$ orthogonal to B_1, it is easy to see that we can take $\lambda = 1$ so that

$$B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

clearly orthogonal to B_1. Similarly, if $B_3 = A_3 - \lambda B_1 - \mu B_2$ is orthogonal to B_1 and B_2, it is easy to see that we can take $\mu = 1$ and $\lambda = 1$ to get

$$B_3 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Notice that $B_3 = A_3 - A_2$. The real reason that this is orthogonal to A_1 and A_2 is that we've arranged that there's no non-zero entry of B_3 corresponding to a non-zero entry of A_1 or A_2 (which have the same span as B_1 and B_2) so it must be orthogonal to them. We could have done this by inspection.

In the same way, we can see that

$$B_4 = A_4 - A_3 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

is orthogonal to all of B_1, B_2 and B_3. Thus we have an orthogonal set

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \qquad B_4 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

and we can make them normal by dividing by their norms:

$$C_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}, \qquad B_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0\end{pmatrix},$$

$$C_3 = \frac{1}{\sqrt{8}}\begin{pmatrix}0&1&1&0\\1&0&0&1\\1&0&0&1\\0&1&1&0\end{pmatrix}, \qquad C_4 = \frac{1}{2}\begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix},$$

as required.

8. Let $v_1 = u_1$. We need $v_2 = u_2 - \lambda v_1$ to be orthogonal to v_1, and $\lambda = \frac{\langle u_2,v_1\rangle}{\langle v_1,v_1\rangle} = \frac{4}{5}$, so that $v_2 = (1/5, 1/5, 1/5, 1/5, -4/5)^T$. Next, we need $v_3 = u_3 - \lambda v_1 - \mu v_2$ to be orthogonal to v_1 and v_2 (or equivalently to u_1 and u_2, since they have the same span). But we know that we should take $\lambda = \frac{\langle u_3,v_1\rangle}{\langle v_1,v_1\rangle} = \frac{3}{5}$, and $\mu = \frac{\langle u_3,v_2\rangle}{\langle v_2,v_2\rangle} = \frac{3/5}{4/5} = \frac{3}{4}$. This gives $v_3 = (1/4, 1/4, 1/4, -3/4, 0)^T$. Next we need $v_4 = u_4 - \lambda v_1 - \mu v_2 - \nu v_3$ to be orthogonal to v_1, v_2 and v_3. But then $\lambda = \frac{\langle u_4,v_1\rangle}{\langle v_1,v_1\rangle} = \frac{2}{5}$, $\mu = \frac{\langle u_4,v_2\rangle}{\langle v_2,v_2\rangle} = \frac{2/5}{4/5} = \frac{1}{2}$, and $\nu = \frac{\langle u_4,v_3\rangle}{\langle v_3,v_3\rangle} = \frac{1/2}{3/4} = \frac{2}{3}$, and so $v_4 = (1/3, 1/3, -2/3, 0, 0)^T$.

   I hope that you spotted the pattern before now, and worked out, just by staring at the vectors, why this had to be the case. So I won't bother to write down the final set of equations; if you do, you will find that $v_5 = (1/2, -1/2, 0, 0, 0)^T$, but you should have been able to predict it anyway from how things have been working so far.

   Of course, these aren't yet orthonormal; we need to divide by the norms of the vectors, to get

$$\begin{aligned}
\hat{v}_1 &= \tfrac{1}{\sqrt{5}}(1,1,1,1,1)^T\\
\hat{v}_2 &= \tfrac{1}{\sqrt{20}}(1,1,1,1,-4)^T\\
\hat{v}_3 &= \tfrac{1}{\sqrt{12}}(1,1,1,-3,0)^T\\
\hat{v}_4 &= \tfrac{1}{\sqrt{6}}(1,1,-2,0,0)^T\\
\hat{v}_5 &= \tfrac{1}{\sqrt{2}}(1,-1,0,0,0)^T.
\end{aligned}$$