Solutions for MAS277 Problems

Solutions for Chapter 1 problems

1. Hopefully you have done this!

2. (a) We need to show that $F = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\} \subseteq \mathbb{R}$ is a subgroup of $\mathbb{R}$ with respect to addition and multiplication. As $\mathbb{R}$ is a group with respect to addition and $\mathbb{R}\backslash\{0\}$ is a group with respect to multiplication we just need to show that F is a subgroup of $\mathbb{R}$ with respect to these operations. We know $H \subseteq G$ is a subgroup of a group G if and only if $g_1 * g_2^{-1} \in H$ whenever $g_1, g_2 \in H$. With respect to addition on F we have $(x + y\sqrt{2})^{-1} = -x - y\sqrt{2}$, and, with respect to multiplication on F, $(x + y\sqrt{2})^{-1} = \frac{1}{x+y\sqrt{2}} = \frac{x}{x^2-2y^2} - \frac{y\sqrt{2}}{x^2-2y^2}$. So for $g_1 = a + b\sqrt{2}$ and $g_2 = c + d\sqrt{2}$ we have:

$$g_1 + (g_2)^{-1} = (a - c) + (b - d)\sqrt{2} \in F,$$

$$g_1(g_2)^{-1} = (a + b\sqrt{2}) \left(\tfrac{c}{c^2-2d^2} - \tfrac{d\sqrt{2}}{c^2-2d^2} \right) = \tfrac{ac-2bd}{c^2-2d^2} + \tfrac{(bc-ad)\sqrt{2}}{c^2-2d^2} \in F.$$

Hence, F is a group with respect to addition (axioms **F1**) and multiplication (axioms **F2**). The distributive law **F3** holds in $\mathbb{R}$ and hence holds in $F \subseteq \mathbb{R}$. Hence, F is a field.

(b) We note that, with respect to multiplication on $F = \{a + bi : a, b \in \mathbb{Z}\}$, we have

$$(a + bi)^{-1} = \tfrac{a}{a^2+b^2} - \tfrac{bi}{a^2+b^2}$$

Setting $a = 3$, $b = 4$ we find that $(3 + 4i)^{-1} = \frac{3}{25} + \frac{4}{25}i \notin F$. Hence, F is not a group with respect to multiplication and so F is not a field.

(c) We have $\sqrt[3]{2} \in F = \{a+b\sqrt[3]{2} : a, b \in \mathbb{Q}\}$. If F is a field then $\sqrt[3]{2}\sqrt[3]{2} = 2^{2/3} \in F$. So, $2^{2/3} = a + b\sqrt[3]{2}$ for some $a, b \in \mathbb{Q}$. Cubing both sides of $2^{2/3} = a + b\sqrt[3]{2}$ we get

$$\begin{aligned} 4 = (2^{2/3})^3 = (a + b\sqrt[3]{2})^3 &= a^3 + 3a^2b\sqrt[3]{2} + 3ab^2 2^{2/3} + 2b^3 \\ &= a^3 + 3a^2b\sqrt[3]{2} + 3ab^2(a + b\sqrt[3]{2}) + 2b^3 \\ \Leftrightarrow \sqrt[3]{2} &= \frac{4 - a^3 - 2b^3 - 3a^2b^2}{3a^2b + 3ab^3} \in \mathbb{Q} \qquad \text{if} \qquad 3a^2b + 3ab^3 \neq 0. \end{aligned}$$

This is a contradiction because we know $\sqrt[3]{2} \notin \mathbb{Q}$. Hence, multiplication on F is not a binary operation (multiplication is not closed) and we conclude that F is not a field.

We now consider the case when $3a^2b + 3ab^3 = 0$. In this case

$$3a^2b + 3ab^3 = 0 \Leftrightarrow 3ab(a + b^2) = 0 \Leftrightarrow a = 0 \text{ or } b = 0 \text{ or } a = -b^2.$$

If $a = 0$ then $\sqrt[3]{2} = b2^{1/3}$ so $b = 2^{\frac{1}{3}}$ a contradiction because $\sqrt[3]{2}$ is irrational and we assumed that $b \in \mathbb{Q}$. If $b = 0$ then $2^{\frac{2}{3}} = a$. Squaring both sides we find $2\sqrt[3]{2} = a^2 \Rightarrow \sqrt[3]{2} = \frac{a^2}{2}$ contradicting our assumption that $\sqrt[3]{2}$ is irrational. From before,

$$4 = (2^{2/3})^3 = (a + b\sqrt[3]{2})^3 = a^3 + 3a^2b\sqrt[3]{2} + 3ab^2(a + b\sqrt[3]{2}) + 2b^3$$

so if $a = -b^2$ then we have

$$2b^6 + 2b^3 - 4 = 0.$$

Treating this as a quadratic equation in b^3, and using the fact that b is rational, we find that $b = 1$. So, $2^{2/3} = -1 + 2^{1/3} < 1$, a contradiction. Hence, F is not closed under multiplication and is therefore not a field.

3. (a) We first show that $0\mathbf{v} = \mathbf{0}$. By **VS1** we have $\mathbf{0} + \mathbf{v} = \mathbf{v} = \mathbf{v} + \mathbf{0}$. We have

$$0\mathbf{v} = (0 + 0)\mathbf{v} = 0\mathbf{v} + 0\mathbf{v} \text{ by } \mathbf{VS5}. \tag{0.1}$$

By **VS1** we have $\mathbf{0} = 0\mathbf{v} - 0\mathbf{v}$. Adding $-0\mathbf{v}$ to both sides of (0.1) we get $\mathbf{0} = 0\mathbf{v}$. So

$$\mathbf{0} = (1 - 1)\mathbf{v} = \mathbf{v} + (-1)\mathbf{v} \Rightarrow (-1)\mathbf{v} = -\mathbf{v}. \tag{0.2}$$

In particular, for $\mathbf{v} = \mathbf{0}$ in we get $(-1)\mathbf{0} = \mathbf{0}$ as claimed.

(b) By **VS1**, $\mathbf{0} + \mathbf{0} = \mathbf{0}$ so

$$\alpha\mathbf{0} = \alpha(\mathbf{0} + \mathbf{0}) = \alpha.\mathbf{0} + \alpha.\mathbf{0} \text{ by } \mathbf{VS4}. \tag{0.3}$$

Adding $-\alpha.\mathbf{0}$ to both sides of (0.3), and using the **VS1** axioms, we get:

$$\mathbf{0} = (\alpha.\mathbf{0} + \alpha.\mathbf{0}) - \alpha.\mathbf{0} = \alpha.\mathbf{0} + (\alpha.\mathbf{0} - \alpha.\mathbf{0})) = \alpha.\mathbf{0} + \mathbf{0} = \alpha.\mathbf{0}.$$

Hence $\mathbf{0} = \alpha.\mathbf{0}$, as required.

(c) By commutativity from **VS1**, we have $\mathbf{y} + (\mathbf{x} - \mathbf{y}) = (\mathbf{x} - \mathbf{y})) + \mathbf{y}$, which equals $\mathbf{x} + (-\mathbf{y} + \mathbf{y})$ from associativity from **VS1**. But $-\mathbf{y} + \mathbf{y} = \mathbf{0}$, so we get $\mathbf{y} + (\mathbf{x} - \mathbf{y}) = \mathbf{x} + \mathbf{0} = \mathbf{x}$ as required.

4. V is not a vector space with this definition as **VS2** fails:

$$1.(x, y) := (x, 0) \neq (x, y), \text{ whenever } y \neq 0.$$

5. (a) This is not closed under scalar multiplication. For example, $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is in V_1, but $(-1)A = \begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix}$ is not. (Once one of the closure rules fails, often many of the "usual rules" fail too, but we won't mention those.)

(b) Again this is not closed under multiplication: $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in V_2$, but $\frac{1}{4}\mathbf{v} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} \notin V_2$.

(c) This time, V_3 is not closed under addition: $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in V_3$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in V_3$, but $\mathbf{u} + \mathbf{v} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \notin V_3$.

(d) This is similar to (c), and is again not closed under addition. Notice that the condition really says "either $p(0) = 0$ or $p(1) = 0$", i.e., p has either 0 or 1 as a root. We can pick a polynomial p with a root at 0, but not at 1, and another polynomial q with a root at 1 but not at 0; then their sum has no root at either 0 or 1. For example, choose $p(x) = x$ and $q(x) = 1 - x$, so that $p(0) = 0$ and $q(1) = 0$. So $p, q \in V_4$. But $p + q$ is the constant polynomial 1, so $p + q \notin V_4$.

6. Clearly $\mathbf{0} = (0, 0, \ldots, 0, \ldots)$ satisfies the recurrence relation.

   Suppose that $\mathbf{a} = (a_0, a_1, a_2, \ldots)$ and $\mathbf{b} = (b_0, b_1, b_2, \ldots)$ are two sequences satisfying the recurrence relation, and let λ be a real number. Then we claim that $\mathbf{a} + \mathbf{b} = (a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots)$ and $\lambda\mathbf{a} = (\lambda a_0, \lambda a_1, \lambda a_2, \ldots)$ also satisfy the recurrence relation. Let's check that, for both these sequences, the $(n+2)$nd term is the sum of α times the $(n+1)$st term and β times the nth term:

$$\begin{aligned} a_{n+2} + b_{n+2} &= (\alpha a_{n+1} + \beta a_n) + (\alpha b_{n+1} + \beta b_n) \\ &= \alpha(a_{n+1} + b_{n+1}) + \beta(a_n + b_n), \end{aligned}$$

   and

$$\begin{aligned} \lambda a_{n+2} &= \lambda(\alpha a_{n+1} + \beta a_n) \\ &= \alpha(\lambda a_{n+1}) + \beta(\lambda a_n), \end{aligned}$$

   as required.

7. U_1 is a subspace: it contains $(0,0,0)^T$ and given $(0, y, z)^T$ and $(0, y', z')^T$, and real numbers α and β, then $\alpha(0, y, z)^T + \beta(0, y', z')^T = (0, \alpha y + \beta y', \alpha z + \beta z')^T$, so the first component is still 0.

   But U_2 is not a subspace: $(0,1,0)^T$ and $(1,0,0)^T$ are in U_2, but their sum is $(1,1,0)^T$, which is not.

   U_3 is a subspace: it contains $(0,0,0)^T$ and given $(x, y, z)^T$ and $(x', y', z')^T$ in U_3, and real numbers α and β, then $\alpha(x, y, z)^T + \beta(x', y', z')^T = (\alpha x + \beta x', \alpha y + \beta y', \alpha z + \beta z')^T$, and the sum of the first two components is

$$(\alpha x + \beta x') + (\alpha y + \beta y') = \alpha(x + y) + \beta(x' + y') = 0,$$

   so any linear combination is again in U_3.

   But U_4 is not a subspace: $(0,1,0)^T$ and $(1,0,0)^T$ are in U_4, but their sum $(1,1,0)^T$ is not in U_4. (Indeed it's enough to note that it does not contain $(0,0,0)^T$.)

8. U_1 is not a subspace, since x^3 and $-x^3$ are both in U_1, but their sum is 0, which does not have degree 3. (Replacing the equals sign with "$\leq$" makes everything fine.)

   U_2 is a subspace: it contains the zero polynomial and if p and q are in U_2, and α and β are real, then $\alpha p + \beta q$ is again in U_2, since

$$(\alpha p + \beta q)(0) = \alpha p(0) + \beta q(0) = \alpha p(1) + \beta q(1) = (\alpha p + \beta q)(1).$$

   U_3 is not a subspace, since if $p(x) = x$, say, so $p \in U_3$, and $\alpha = -1$, say, then $\alpha p(x) = -x$, and this is not in U_3.

9. Let's take $(w, x, y, z)^T \in U \cap V$. This means that

$$\begin{aligned} w - x + y - z &= 0 \\ w + x + y &= 0 \\ x + y + z &= 0. \end{aligned}$$

The second and third imply that $w = z$; substituting into the first implies that $x = y$; substituting back into the second implies that $w + 2x = 0$. We find that $(w, x, y, z)^T = (-2x, x, x, -2x)^T$. So

$$U \cap V = \{(-2x, x, x, -2x)^T \mid x \in \mathbb{R}\}.$$

Now let's work out $U \cap W$. We need to work out which vectors $(u, u+v, u+2v, u+3v)^T$ lie in U, i.e., where $u - (u + v) + (u + 2v) - (u + 3v) = 0$, i.e., $-2v = 0$, so that $v = 0$. It follows that

$$U \cap W = \{(u, u, u, u)^T \mid u \in \mathbb{R}\}.$$

Finally, we'll work out $V \cap W$. We need to work out which vectors $(u, u + v, u + 2v, u + 3v)^T$ lie in V. This requires

$$u + (u + v) + (u + 2v) = 0 = (u + v) + (u + 2v) + (u + 3v).$$

Combining these, we see that $u = v = 0$, so that $V \cap W = \{0\}$.

10. Usually, two planes will intersect in a line, and this line in turn will intersect a third plane at a point (the origin). So the "typical" intersection of three planes through the origin just contains the origin itself. However,

$$P \cap Q \cap R = \{(x, -2x, x)^T \mid x \in \mathbb{R}\},$$

so in this example, the three planes intersect in the line of multiples of $(1, -2, 1)$.

11. U_1 is a subspace: it contains $(0, 0, 0, 0)^T$ and given $(w, x, y, z)^T$ and $(w', x', y', z')^T$, and real numbers α and β, then $\alpha(w, x, y, z)^T + \beta(w', x', y', z')^T = (\alpha w + \beta w', \alpha x + \beta x', \alpha y + \beta y', \alpha z + \beta z')^T$, and

$$(\alpha w + \beta w') + (\alpha x + \beta x') = \alpha(w + x) + \beta(w' + x') = 0,$$

so this linear combination is again in U_1.

But U_2 is not a subspace: $(0, 1, 0, 0)^T$ and $(1, 0, 0, 0)^T$ are in U_2, but their sum is $(1, 1, 0, 0)^T$, which is not. (Or one could just note that it doesn't contain $(0, 0, 0, 0)^T$.)

U_3 is a subspace: it contains $(0, 0, 0, 0)^T$ and given $(w, x, y, z)^T$ and $(w', x', y', z')^T$, and real numbers α and β, then $\alpha(w, x, y, z)^T + \beta(w', x', y', z')^T = (\alpha w + \beta w', \alpha x + \beta x', \alpha y + \beta y', \alpha z + \beta z')^T$, and

$$\begin{aligned}(\alpha w + \beta w') &+ 2(\alpha x + \beta x') + 3(\alpha y + \beta y') + 4(\alpha z + \beta z')\\ &= \alpha(w + 2x + 3y + 4z) + \beta(w' + 2x' + 3y' + 4z') = 0,\end{aligned}$$

so this linear combination is again in U_3.

U_4 is not a subspace: $(-1, 1, 0, 0)^T \in U_4$, but if we scale this by 2, to get $(-2, 2, 0, 0)^T$, this is not in U_4.

U_5 is a subspace (it is rather unusual to see conditions which aren't linear which do specify a subspace!) – but $w^2 + x^2 = 0$ implies that $w = x = 0$ since w and x are real. So U_5 is also specified by the condition $U_5 = \{(w, x, y, z)^T \mid w = x = 0\}$, and this is easily seen to be a subspace.

12. U_1 is a subspace: it contains the zero function and if $f, g \in U_1$ and $\alpha, \beta \in \mathbb{R}$, then $(\alpha f + \beta g)(0) = \alpha f(0) + \beta g(0) = 0$, so $\alpha f + \beta g \in U_1$.

    U_2 is not: if $f \in U_2$, then $2f \notin U_2$.

    U_3 is also not a subspace: if $f \in U_3$, and $f(0) > 0$ (e.g., $f(x) = 1$ for all x) then $(-1)f \notin U_3$.

    U_4 is a subspace: it contains the zero function and if $f, g \in U_1$ and $\alpha, \beta \in \mathbb{R}$, then $(\alpha f + \beta g)(0) = \alpha f(0) + \beta g(0) = \alpha f(1) + \beta g(1) = (\alpha f + \beta g)(1)$, so $\alpha f + \beta g \in U_4$.

    Finally, U_5 is not a subspace. Let $f(x) = x(x-2)$ and $g(x) = 1$. Then $f(0) = f(2) = 0$, so $f(0)f(1) = 0 = f(2)f(3)$, and $g(0) = g(1) = g(2) = g(3) = 1$ so $g(0)g(1) = 1 = g(2)g(3)$. So $f, g \in U_5$. But $f + g = x^2 - 2x + 1 = (x-1)^2$, so $(f+g)(0)(f+g)(1) = 0$ and $(f+g)(2)(f+g)(3) = 2$, so $f + g \notin U_5$.

13. There are plenty of possible answers. Here are some:

    (a) $W = \mathbb{R}[x]_{\leq 2}$;

    (b) $W = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \in M_{2\times 3}(\mathbb{R}) \;\middle|\; z \in \mathbb{R} \right\}$;

    (c) $W = \{(x, -2x, x)^T \in \mathbb{R}^3 \,|\, x \in \mathbb{R}\}$.

14. There are plenty of possible answers. Here are some:

    (a) $U = \{(w, x, 0, 0)^T \in \mathbb{R}^4 \,|\, w, x \in \mathbb{R}\}$, $W = \{(0, 0, y, z)^T \in \mathbb{R}^4 \,|\, y, z \in \mathbb{R}\}$;

    (b) $U = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) \;\middle|\; a, b \in \mathbb{R} \right\}$, $W = \left\{ \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \;\middle|\; c, d \in \mathbb{R} \right\}$;

    (c) $U = \{(x, -2x, x)^T \in \mathbb{R}^3 \,|\, x \in \mathbb{R}\}$, $W = \{(-2w, w, w)^T \in \mathbb{R}^3 \,|\, w \in \mathbb{R}\}$.

15. Since $U \cap V = U \cap W = \{0\}$, we have $(U \cap V) + (U \cap W) = \{0\}$. On the other hand, it is easy to see that any vector in $\mathbb{R}^2$ can be written as the sum of a vector in V and a vector in W (indeed, $(x, y)^T = (x-y, 0)^T + (y, y)^T$). So $U \cap (V + W) = U \cap \mathbb{R}^2 = U$. Thus the left-hand side is U, and the right-hand side is $\{0\}$, so they are different.

16. Let $f(x) = ax^2 + bx + c$ be a polynomial in $U \cap W$. As $f(0) = 0$, we see that $c = 0$. Also, $f \in W$, so that $f(1) + f(-1) = 0$, and $f(1) + f(-1) = 2a + 2c = 0$. As $c = 0$, we see that $a = 0$, and so $f(x) = bx$.

    On the other hand, a general polynomial in V can be written as a sum of something in U and something in W:

    $$ax^2 + bx + c = ((a+c)x^2 + bx) + (-cx^2 + c);$$

    the first polynomial has 0 constant term, so lies in U; the second has $f(1) = f(-1) = 0$, so certainly lies in W. Therefore $V = U + W$.

17. Suppose that $(x, y)^T \in L \cap M$. Then $(x, y)^T = (s, 2s)^T = (2t, t)^T$ for some $s, t \in \mathbb{R}$. Then $s = 2t$ and $2s = t$, so $s = t = 0$. Thus $(x, y)^T = (0, 0)^T$ and $L \cap M = 0$.

    Take a general vector $(a, b)^T \in \mathbb{R}^2$. We need to find s and t so that

    $$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} s \\ 2s \end{pmatrix} + \begin{pmatrix} 2t \\ t \end{pmatrix}.$$

Solving the two equations $s+2t=a$, $2s+t=b$ simultaneously, we get $s=(2b-a)/3$ and $t=(2a-b)/3$. So

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2b-a \\ 2(2b-a) \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 2(2a-b) \\ 2a-b \end{pmatrix} \in L+M.$$

18. To see that $U \cap W = 0$, let $A \in U \cap W$. Then since $A \in U$, we know that $A^T = A$, and since $A \in W$, we know that $A^T = -A$. Therefore $A=-A$, and so $A=0$.

There is a slick way to see that $V = U + W$: given any matrix A, we write

$$A = \frac{A+A^T}{2} + \frac{A-A^T}{2}.$$

The first matrix $\frac{1}{2}(A+A^T)$ lies in U, since its transpose is $\frac{1}{2}(A^T+A^{TT}) = \frac{1}{2}(A^T+A)$, which is the matrix itself. Similarly, the second matrix lies in W:

$$\left(\frac{A-A^T}{2}\right)^T = \left(\frac{A^T-A^{TT}}{2}\right) = \left(\frac{A^T-A}{2}\right) = -\left(\frac{A-A^T}{2}\right),$$

as required. (Of course, the same argument applies to $M_{n\times n}(\mathbb{R})$ for any n.)

Without the slick method, it's likely that you will argue as follows:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & \frac{b+d}{2} & \frac{c+g}{2} \\ \frac{b+d}{2} & e & \frac{f+h}{2} \\ \frac{c+g}{2} & \frac{f+h}{2} & i \end{pmatrix} + \begin{pmatrix} 0 & \frac{b-d}{2} & \frac{c-g}{2} \\ \frac{d-b}{2} & 0 & \frac{f-h}{2} \\ \frac{g-c}{2} & \frac{h-f}{2} & 0 \end{pmatrix},$$

and the first matrix is clearly symmetric, so is in U, and the second is antisymmetric, so is in W.

19. (a) These are not linearly independent (e.g., $\mathbf{u}_1 + \mathbf{u}_4 = \mathbf{u}_2 + \mathbf{u}_3$), nor do they span $\mathbb{R}^3$, since there is no way that we can make $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ or any other vector where the middle component is non-zero.

 (b) These are not linearly independent (e.g., $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = 2\mathbf{u}_4$), but they do span; we can write each vector in the standard basis in terms of these $\mathbf{v}_i$ – since $\mathbf{e}_1 = \mathbf{v}_4 - \mathbf{v}_3$, $\mathbf{e}_2 = \mathbf{v}_4 - \mathbf{v}_2$ and $\mathbf{e}_3 = \mathbf{v}_4 - \mathbf{v}_1$.

 (c) These are linearly independent, but do not span – the middle component is the average of the other two in both vectors, and this will hold for any linear combination, so we can't make $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ or any other vector where the middle component is not the average of the top and bottom ones.

 (d) These are linearly independent – in any linear combination making the zero vector,

$$\lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

 it is easy to see that each $\lambda_i = 0$ (first, $\lambda_1 = 0$ by considering the top component, then $\lambda_2 = 0$ by looking at the middle component, and then $\lambda_3 = 0$). They also span – we can write each element of the standard basis in terms of this set: $\mathbf{e}_1 = \mathbf{x}_1 - \frac{1}{2}\mathbf{x}_2$, $\mathbf{e}_2 = \frac{1}{2}\mathbf{x}_2 - \frac{1}{4}\mathbf{x}_3$, and $\mathbf{e}_3 = \frac{1}{4}\mathbf{x}_3$.

20. (a) These are linearly independent. Given a linear combination making the zero vector, considering the first component shows that the coefficient of $\mathbf{u}_1$ is zero; then the second component gives that the coefficient of $\mathbf{u}_2$ is zero, and finally the coefficient of $\mathbf{u}_3$ is zero.

    (b) These are not linearly independent: adding $\mathbf{v}_2$ to $\frac{1}{2}\mathbf{v}_3$ is the same as $2\mathbf{v}_1$. Thus

    $$4\mathbf{v}_1 - 2\mathbf{v}_2 - \mathbf{v}_3 = 0.$$

    (c) These are linearly independent: we can write the standard basis in terms of these vectors: $\mathbf{e}_3 = \mathbf{w}_1 - \mathbf{w}_3$, $\mathbf{e}_2 = \mathbf{w}_2 - \mathbf{w}_3 - 3\mathbf{w}_1$, and then $\mathbf{e}_1 = \mathbf{w}_3 - \mathbf{e}_3 - \mathbf{e}_2$, and we've already written $\mathbf{e}_2$ and $\mathbf{e}_3$ in terms of the $\mathbf{w}_i$.

21. This is more or less immediate. α and β are linearly dependent if and only if there are rationals λ and μ, not both 0, such that $\lambda\alpha + \mu\beta = 0$. Notice that $\beta \neq 0$, so $\lambda \neq 0$. Then rearranging gives $\alpha/\beta = -\mu/\lambda$.

22. We could take $(1,1,0,0)^T$, $(0,0,1,1)^T$, $(1,0,0,0)^T$ and $(0,0,1,0)^T$ as our first basis, and $(1,1,0,0)^T$, $(0,0,1,1)^T$, $(0,1,0,0)^T$ and $(0,0,0,1)^T$ as our second basis. In both cases, it is easy to see that all the standard basis elements can be written in terms of these two sets, so both are bases, and clearly they satisfy the conditions of the question.

23. (a) This does not span. There is no way that any linear combination of $\mathcal{A}$ can have non-zero top left entry, so the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ cannot be expressed as a linear combination.

    (b) This does not span. It is clear that each entry in $\mathcal{B}$ has their top right entry equal to the bottom left entry. Any linear combination will have the same property, so $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ cannot be expressed as a linear combination of elements of $\mathcal{B}$.

    (c) This set does span. There is an obvious "standard" basis (although we have only used the terminology for a basis of $\mathbb{R}^n$), and this is formed by the first matrix in $\mathcal{C}$, the second minus the first, the third minus the second, and the fourth minus the third.

    (d) This set does not span. Each matrix in $\mathcal{D}$ has trace zero, i.e., the sum of the two diagonal entries is 0. Any linear combination of these matrices will also have this property, and so $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ cannot be expressed as a linear combination of elements of $\mathcal{D}$.

24. We need to write an arbitrary polynomial $ax^2 + bx + c$ as a linear combination of x^2, $(x+1)^2 = x^2 + 2x + 1$, $(x+2)^2 = x^2 + 4x + 4$; i.e.,

    $$ax^2 + bx + c = px^2 + q(x^2 + 2x + 1) + r(x^2 + 4x + 4),$$

    or

    $$\begin{aligned} p + q + r &= a \\ 2q + 4r &= b \\ q + 4r &= c. \end{aligned}$$

Subtracting the third from the second gives $q = b - c$, so then $4r = 2c - b$ from either of the last two equations, and then the first equation gives $p = a - q - r = a - (b - c) - (2c - b)/4 = a - 3b/4 + c/2$, and so we can indeed write any quadratic as a linear combination of s_0, s_1, s_2.

(Alternatively, we can write each of $1, x, x^2$ as a linear combination of s_0, s_1 and s_2, or we can show that s_0, s_1 and s_2 are linearly independent; by a theorem in the course, they form a basis of the 3-dimensional space $\mathbb{R}[x]_{\leq 2}$, so span.)

25. Suppose there is some linear relation $\lambda f + \mu g + \nu h = 0$. The left-hand side is a function in $C(\mathbb{R})$, so we can evaluate it at any point. Let's first evaluate it at a:

$$\lambda f(a) + \mu g(a) + \nu h(a) = 0,$$

and substituting in the values, we see that this gives $\lambda = 0$. If we use $x = b$ instead, we see that $\mu = 0$, and using $x = c$ gives $\nu = 0$. So there are no non-trivial linear relations.

26. (a) We have the formula

$$\sin(x + \alpha) = \cos\alpha \sin x + \sin\alpha \cos x,$$

and this is a non-trivial linear relation involving the three given functions.

(b) Given a linear relation

$$\lambda \sin x + \mu \cos x + \nu \sin kx,$$

we can put $x = 0$, to see that $\mu = 0$. Then $\lambda \sin x + \nu \sin kx = 0$. If $k > 1$, we can put $x = \frac{\pi}{k}$, so that $\sin x \neq 0$ but $\sin kx = 0$. Using this value of x shows that $\lambda = 0$, and then we see that $\nu = 0$.

27. Take a general polynomial $f(x) = ax^3 + bx^2 + cx + d$. Then $f \in V$ if $f(x) + f(-x) = 0$. But $f(x) + f(-x) = 2bx^2 + 2d$, and this is identically zero if $b = d = 0$. Thus the general form of a polynomial in V is $ax^3 + cx$, so a basis is given by $\{x^3, x\}$.

Similarly, $f \in W$ if $f''(1) = 2f'(1) = 6f(1)$. But

$$\begin{aligned} f'(x) &= 3ax^2 + 2bx + c \\ f''(x) &= 6ax + 2b \end{aligned}$$

so $f''(1) = 6a + 2b$, $2f'(1) = 6a + 4b + 2c$ and $6f(1) = 6a + 6b + 6c + 6d$. These are equal if

$$6a + 2b = 6a + 4b + 2c = 6a + 6b + 6c + 6d,$$

and then a is arbitrary, $2b + 2c = 0$ and $2b + 4c + 6d = 0$. So $c = -b$ and $d = b/3$. Then $f(x) = ax^3 + bx^2 - bx + \frac{b}{3}$, so a basis is $\{x^3, x^2 - x + \frac{1}{3}\}$.

Clearly an element $f(x) = ax^3 + bx^2 - bx + \frac{b}{3}$ is also in V only if $b = 0$, so that $V \cap W$ consists of polynomials ax^3, so a basis is $\{x^3\}$.

A basis for $V + W$ is got by taking the union of the basis of V and the basis of W, so a basis is $\{x^3, x, x^2 - x + \frac{1}{3}\}$. A general polynomial in $V + W$ is therefore of the form $f(x) = ax^3 + bx^2 + cx + \frac{b}{3}$. Let's check that this is the general form of a polynomial satisfying $f''(0) = 6f(0)$. Indeed, given a polynomial $ax^3 + bx^2 + cx + d$, we have $f''(0) = 2b$ and $f(0) = d$. So $f''(0) = 6f(0)$ precisely when $2b = 6d$, i.e., when $d = \frac{b}{3}$. So this means that the general form of a polynomial satisfying $f''(0) = 6f(0)$ is $ax^3 + bx^2 + cx + \frac{b}{3}$, as required.

28. Let's start by finding a basis for $U \cap W$. It is clear that a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in U precisely when $b = c$, and is in W when $a + d = 0$. So the intersection is all matrices of the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, and a basis is therefore $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$.

    Since U is the set of matrices of the form $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$, this has dimension 3, so we expect to add just one more matrix to our basis for $U \cap W$ to get a basis for U; any matrix in U but not W will do; for example, we can take $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ as a basis for U.

    Similarly, W is the set of matrices of the form $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$; this also has dimension 3, so we expect to add just one matrix to our basis for $U \cap W$ to get a basis for W; any matrix in W but not in U will do; for example, we can take $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ as a basis for W.

    Then $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ is a basis for $M_{2\times 2}(\mathbb{R})$.

29. We have

$$\dim(U + V) = \dim U + \dim V - \dim(U \cap V) = 2 + 3 - 1 = 4.$$

    Similarly,

$$\dim(V + W) = \dim V + \dim W - \dim(V \cap W) = 3 + 4 - 2 = 5,$$

    and

$$\dim(U + V + W) = \dim(U + V) + \dim W - \dim((U + V) \cap W) = 4 + 4 - 3 = 5.$$

    Now $V + W$ is a subspace of $U + V + W$, but they have the same dimension, and so it follows that they must be equal. Therefore any vector in $U + V + W$ can be written as the sum of a vector in V and a vector in W. In particular, take a vector in U. Then this vector lies in $U + V + W$, and therefore in $V + W$, and we conclude that U is a subspace of $V + W$, as required.

30. Every complex number is a real number times 1 added to a real number times i (or, in other words, may be written as $a + bi$, with $a, b \in \mathbb{R}$). It follows that $\mathbb{C}$ has dimension 2 as a real vector space (with basis $\{1, i\}$).

    As a complex vector space, however, the dimension is 1: every complex number is a complex number times 1, so the basis is $\{1\}$, and we have a 1-dimensional space.

31. (a) Suppose that each row of the matrix adds to a constant R, and that each column adds to a constant C – it is easy to see that $R = C$, since the sum of all the entries in the matrix is $3R$, as the entries can be added in the three rows, and is also $3C$, as the entries can be added in the three columns. [The wording of the question is a bit ambiguous – you probably assumed $R = C$ from the wording, which is fine, but I thought I'd point out anyway that this really is necessary!]

V is a subspace: two semi-magic squares add to be a semi-magic square, and if we scale a semi-magic square by any real number, it is again semi-magic.

Let's note that if we specify the first 5 entries of a semi-magic square, the remainder can be filled in unambiguously; beginning with

$$\begin{pmatrix} a & b & c \\ d & e & \\ & & \end{pmatrix},$$

we fill in the remaining rows and columns assuming a common row and column sum of $a+b+c$:

$$\begin{pmatrix} a & b & c \\ d & e & a+b+c-d-e \\ b+c-d & a+c-e & -c+d+e \end{pmatrix}.$$

We can write this as:

$$a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}+b\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}+c\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}+d\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}+e\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

so these five matrices form a basis, and the space has dimension 5.

(b) In fact, none of the basis elements are magic, since in no case do the diagonals both have the same sum as the rows and columns.

(c) Again, the sum of two nearly magic squares is nearly magic, and the scalar multiple of a nearly magic square is also nearly magic, so U is a subspace. To work out a basis of U, let's note that an element of V is nearly magic if the two diagonal sums are the same:

$$a+e+(-c+d+e) = c+e+(b+c-d),$$

so that $e=-a+b+3c-2d$, and the matrix is then

$$\begin{pmatrix} a & b & c \\ d & -a+b+3c-2d & 2a-2c+d \\ b+c-d & 2a-b-2c+2d & -a+b+2c-d \end{pmatrix}.$$

We can write this as

$$a\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 2 & -1 \end{pmatrix}+b\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}+c\begin{pmatrix} 0 & 0 & 1 \\ 0 & 3 & -2 \\ 1 & -2 & 2 \end{pmatrix}+d\begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ -1 & 2 & -1 \end{pmatrix},$$

so these four matrices form a basis, and so $\dim U = 4$. Since the dimension of U is greater than the dimension of magic squares, there are nearly magic squares which are not magic - and each of the four matrices in the above basis are nearly magic, but not magic.

32. We start by forming the matrix whose columns are the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, and then the desired matrix is its inverse:

$$\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}.$$

Explicitly,

$$\begin{pmatrix} a \\ b \end{pmatrix} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 = \begin{pmatrix} 3\alpha_1 + 2\alpha_2 \\ 2\alpha_1 + \alpha_2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix},$$

and the result follows by multiplying both sides by the inverse of the matrix.

33. We form $A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$. Then we know that

$$\begin{pmatrix} a \\ b \end{pmatrix} = A_1 \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = A_2 \begin{pmatrix} \alpha'_1 \\ \alpha'_2 \end{pmatrix},$$

so the desired matrix is

$$A = A_1^{-1} A_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}.$$

Explicitly, $\alpha'_1 \mathbf{v}_3 + \alpha'_2 \mathbf{v}_4 = \begin{pmatrix} \alpha'_2 \\ \alpha'_1 - \alpha'_2 \end{pmatrix} = (-\alpha'_1 + 2\alpha'_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (\alpha'_1 - \alpha'_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so $\alpha_1 = -\alpha'_1 + 2\alpha'_2$ and $\alpha_2 = \alpha'_1 - \alpha'_2$, i.e., $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha'_1 \\ \alpha'_2 \end{pmatrix}$ as expected.

34. We form $A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. As above, we put

$$A = A_1^{-1} A_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Explicitly, $\alpha'_1 \mathbf{v}_2 + \alpha'_2 \mathbf{v}_3 = \begin{pmatrix} \alpha'_1 \\ \alpha'_1 + \alpha'_2 \end{pmatrix} = (-\alpha'_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (\alpha'_1 + \alpha'_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so $\alpha_1 = -\alpha'_2$ and $\alpha_2 = \alpha'_1 + \alpha'_2$, i.e., $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha'_1 \\ \alpha'_2 \end{pmatrix}$ as expected.

Solutions for Chapter 2 problems

1. (a) This is linear: $\phi_1(\alpha(x,y)^T+\beta(x',y')^T)=\phi_1((\alpha x+\beta x',\alpha y+\beta y'))=\alpha x+\beta x'$, but $\phi_1((x,y)^T)=x$, and $\phi_1((x',y')^T)=x'$, so $\phi_1(\alpha(x,y)^T+\beta(x',y')^T)=\alpha\phi_1((x,y)^T)+\beta\phi_1((x',y')^T)$.

   (b) This is not linear: $\phi_2((1,1)^T)=2$, but $\phi_2((2,2)^T)=3\neq 2\phi_2((1,1)^T)=4$.

   (c) This is not linear: $\phi_3((1,1)^T)=1$, but $\phi_3((2,2)^T)=4\neq 2\phi_3((1,1)^T)=2$.

   (d) This is linear:

$$\begin{aligned}\phi_4(\alpha(x,y)^T+\beta(x',y')^T) &= \phi_4((\alpha x+\beta x',\alpha y+\beta y')^T)\\ &= (\alpha x+\beta x')+2(\alpha y+\beta y')\\ &= \alpha(x+2y)+\beta(x'+2y')\\ &= \alpha\phi_4((x,y)^T)+\beta\phi_4((x',y')^T),\end{aligned}$$

   as required.

   (e) This is not linear: $\phi_5((1,1)^T)=\phi_5((1,-1)^T)=\sqrt{2}$, but $\phi_5((1,1)^T+(1,-1)^T)=\phi_5((2,0)^T)=2\neq 2\sqrt{2}$.

2. (a) This is linear:

$$\begin{aligned}\phi_1(\alpha(x,y,z)^T+\beta(x',y',z')^T) &= \phi_1((\alpha x+\beta x',\alpha y+\beta y',\alpha z+\beta z')^T)\\ &= (\alpha x+\beta x')+(\alpha y+\beta y')\\ &= \alpha(x+y)+\beta(x'+y')\\ &= \alpha\phi_1((x,y,z)^T)+\beta\phi_1((x',y',z')^T),\end{aligned}$$

   as required.

   (b) This is not linear: $\phi_2((1,1,1)^T)=0$ but $\phi_2((2,2,2)^T)=-2\neq 2\phi_2((1,1,1)^T)$.

   (c) This is not linear: $\phi_3((1,1,1)^T)=0$ but $\phi_3((2,2,2)^T)=1\neq 2\phi_3((1,1,1)^T)$.

   (d) This is not linear: $\phi_4((1,1,1)^T)=1$ but $\phi_4((2,2,2)^T)=8\neq 2\phi_4((1,1,1)^T)$.

   (e) This is linear:

$$\begin{aligned}\phi_5(\alpha(x,y,z)^T+\beta(x',y',z')^T) &= \phi_5((\alpha x+\beta x',\alpha y+\beta y',\alpha z+\beta z')^T)\\ &= (\alpha x+\beta x')-2(\alpha y+\beta y')+3(\alpha z+\beta z')\\ &= \alpha(x-2y+3z)+\beta(x'-2y'+3z')\\ &= \alpha\phi_5((x,y,z)^T)+\beta\phi_5((x',y',z')^T),\end{aligned}$$

   as required.

3. (a) This is linear:

$$\begin{aligned}\phi_1(\alpha f+\beta g) &= \int_{-1}^{2}(\alpha f+\beta g)(x)\,dx=\int_{-1}^{2}\alpha f(x)+\beta g(x)\,dx\\ &= \alpha\int_{-1}^{2}f(x)\,dx+\beta\int_{-1}^{2}g(x)\,dx=\alpha\phi_1(f)+\beta\phi_1(g)\end{aligned}$$

   as required.

(b) This is not linear: if $f(x) = 1$ for all x, $\phi_2(f) = 2$, but $\phi_2(2f) = 8 \neq 2\phi_2(f)$.

(c) This is linear:

$$\begin{aligned}\phi_3(\alpha f + \beta g) &= \int_0^1 x(\alpha f + \beta g)(x)\,dx = \int_0^1 \alpha x f(x) + \beta x g(x)\,dx \\ &= \alpha \int_0^1 x f(x)\,dx + \beta \int_0^1 x g(x)\,dx = \alpha\phi_3(f) + \beta\phi_3(g)\end{aligned}$$

as required.

(d) This is linear:

$$\begin{aligned}\phi_4(\alpha f + \beta g) &= \int_0^1 (\alpha f + \beta g)(x^2)\,dx = \int_0^1 \alpha f(x^2) + \beta g(x^2)\,dx \\ &= \alpha \int_0^1 f(x^2)\,dx + \beta \int_0^1 g(x^2)\,dx = \alpha\phi_4(f) + \beta\phi_4(g)\end{aligned}$$

as required.

(e) This is linear:

$$\begin{aligned}\phi_5(\alpha f + \beta g) &= (\alpha f + \beta g)(0) + (\alpha f + \beta g)'(1) + (\alpha f + \beta g)''(2) \\ &= \alpha(f(0) + f'(1) + f''(2)) + \beta(g(0) + g'(1) + g''(2)) \\ &= \alpha\phi_5(f) + \beta\phi_5(g)\end{aligned}$$

as required.

(f) This is not linear: if $f(x) = x$ and $g(x) = 1 - x$, then $f(0) = g(1) = 0$, so $f(0)f(1) = g(0)g(1) = 0$. But $(f+g)(x) = 1$ for all x, and so $\phi_6(f+g) = 1$, but $\phi_6(f) = \phi_6(g) = 0$.

4. Yes:

$$\begin{aligned}\phi(\alpha(x,y)^T + \beta(x',y')^T) &= \phi((\alpha x + \beta x', \alpha y + \beta y')^T) \\ &= ((\alpha x + \beta x') + (\alpha y + \beta y'), ((\alpha x + \beta x') - (\alpha y + \beta y')^T) \\ &= \alpha(x+y, x-y)^T + \beta(x'+y', x'-y')^T \\ &= \alpha\phi((x,y)^T) + \beta\phi((x',y')^T),\end{aligned}$$

as required.

5. As before,

$$\begin{aligned}\phi(\alpha f + \beta g)(x) &= (\alpha f + \beta g)(x+1) \\ &= \alpha f(x+1) + \beta g(x+1) \\ &= \alpha\phi(f)(x) + \beta\phi(g)(x);\end{aligned}$$

as this holds for all x, $\phi(\alpha f + \beta g) = \alpha\phi(f) + \beta\phi(g)$.

6. There are many different answers. Here are some examples which work:

   (a) $\phi((x,y,z,w)^T) = (x,y)^T$;

   (b) $\phi(A) = (A_{11}, A_{12})^T$;

   (c) $\phi(A) = A_{11}x + A_{12}$;

   (d) $\phi(a_d x^d + \cdots + a_0) = \begin{pmatrix} a_0 & 0 \\ 0 & 0 \end{pmatrix}$.

7. A typical element of $\mathbb{R}[x]_{\leq 1}$ is a polynomial $ax+b$. We can regard this as $a.x+b.1$, where $\{x,1\}$ is a basis for $\mathbb{R}[x]_{\leq 1}$. By linearity,

$$\phi(ax+b) = a\phi(x) + b\phi(1);$$

   we simply put $u = \phi(x)$ and $v = \phi(1)$, and the result follows.

8. (a) If $v_1, \ldots, v_n$ are linearly dependent, there will be some coefficients $\lambda_1, \ldots, \lambda_n$, not all zero, such that $\lambda_1 v_1 + \cdots + \lambda_n v_n$. Now apply ϕ; by linearity:

$$\begin{aligned} \lambda_1\phi(v_1) + \cdots + \lambda_n\phi(v_n) &= \phi(\lambda_1 v_1 + \cdots + \lambda_n v_n) \\ &= \phi(0) = 0, \end{aligned}$$

   and this is a non-trivial linear relation between $\phi(v_1), \ldots, \phi(v_n)$, as required.

   (b) For example, let $V = W = \mathbb{R}^2$, and suppose that $\phi((x,y)^T) = (x+y,0)^T$. Then $v_1 = (1,0)^T$ and $v_2 = (0,1)^T$ are linearly independent, but $\phi((1,0)^T) = \phi((0,1)^T) = (1,0)^T$, so $\phi(v_1)$ and $\phi(v_2)$ are dependent.

   (c) If not, then they are dependent – but then the first part shows that $\phi(v_1), \ldots, \phi(v_n)$ would be dependent, and our hypothesis is that they are not (this is just the contrapositive of the first one!).

9. Suppose that $\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a+d$. With the given form of the map, we have

$$\phi\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} p & q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = pr.$$

   We therefore require $pr = 1$. Similarly,

$$\phi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} p & q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = ps,$$

   and so $ps = 0$. But also

$$\phi\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} p & q \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = qs,$$

   and so $qs = 1$. These three equations cannot be satisfied simultaneously: if $pr = qs = 1$, then clearly none of p, q, r or s can be zero, and so we cannot have $ps = 0$. So we have a contradiction.

10. We have

$$\begin{aligned}\phi(1) &= 1\\ \phi(x) &= x+1\\ \phi(x^2) &= x^2+2x+2\end{aligned}$$

so that

$$\begin{bmatrix}1\\0\\0\end{bmatrix}\mapsto\begin{bmatrix}1\\0\\0\end{bmatrix},\qquad \begin{bmatrix}0\\1\\0\end{bmatrix}\mapsto\begin{bmatrix}1\\1\\0\end{bmatrix},\qquad \begin{bmatrix}0\\0\\1\end{bmatrix}\mapsto\begin{bmatrix}2\\2\\1\end{bmatrix},$$

where we write $\begin{bmatrix}a\\b\\c\end{bmatrix}$ to stand for $a+bx+cx^2$, and then the matrix is $\begin{pmatrix}1&1&2\\0&1&2\\0&0&1\end{pmatrix}$, with trace 3 and determinant 1. (Indeed, all the eigenvalues are 1.)

11. From the formula,

$$\begin{pmatrix}1\\0\\0\end{pmatrix}\mapsto\begin{pmatrix}0\\1\\1\end{pmatrix},\qquad \begin{pmatrix}0\\1\\0\end{pmatrix}\mapsto\begin{pmatrix}1\\0\\1\end{pmatrix},\qquad \begin{pmatrix}0\\0\\1\end{pmatrix}\mapsto\begin{pmatrix}1\\1\\0\end{pmatrix},$$

so the matrix is $P=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$.

With respect to the new basis, we observe that

$$\begin{aligned}\phi(\mathbf{u}_1) &= \phi\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)=\begin{pmatrix}2\\2\\2\end{pmatrix}=2\mathbf{u}_1\\ \phi(\mathbf{u}_2) &= \phi\left(\begin{pmatrix}1\\-1\\0\end{pmatrix}\right)=\begin{pmatrix}-1\\1\\0\end{pmatrix}=-\mathbf{u}_2\\ \phi(\mathbf{u}_3) &= \phi\left(\begin{pmatrix}0\\1\\-1\end{pmatrix}\right)=\begin{pmatrix}0\\-1\\1\end{pmatrix}=-\mathbf{u}_3,\end{aligned}$$

so that the matrix is just $Q=\begin{pmatrix}2&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$.

We can also see this from the change-of-basis theory. Let A denote the matrix whose columns are the $\mathbf{u}_i$, so $A=\begin{pmatrix}1&1&0\\1&-1&1\\1&0&-1\end{pmatrix}$. So $A(\mathbf{e}_i)=\mathbf{u}_i$. To get the matrix of the map with respect to the $\mathbf{u}_i$, we need to work out $\phi(\mathbf{u}_i)$ in terms of the $\mathbf{u}_i$. But we convert to the standard basis with A. Then apply the matrix with respect to the standard basis above, and then apply A^{-1} to get back to the basis $\{\mathbf{u}_i\}$. That is, we expect that $Q=A^{-1}PA$, and this is easily calculated to be as above.

12. We have

$$\begin{aligned}\phi(1) &= (1,0,0)^T = \mathbf{e}_1\\ \phi(x) &= (0,1,0)^T = \mathbf{e}_2\\ \phi(x^2) &= (0,2,2)^T = 2\mathbf{e}_2 + 2\mathbf{e}_3\end{aligned}$$

i.e.,

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \mapsto \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad \begin{bmatrix}0\\1\\0\end{bmatrix} \mapsto \begin{bmatrix}0\\1\\0\end{bmatrix}, \qquad \begin{bmatrix}0\\0\\1\end{bmatrix} \mapsto \begin{bmatrix}0\\2\\2\end{bmatrix},$$

where $\begin{bmatrix}a\\b\\c\end{bmatrix}$ stands for $a + bx + cx^2$, so the matrix is $\begin{pmatrix}1&0&0\\0&1&2\\0&0&2\end{pmatrix}$.

13. We have

$$\kappa(\mathbf{e}_1) = a_1; \qquad \kappa(\mathbf{e}_2) = a_2; \qquad \kappa(\mathbf{e}_3) = a_3,$$

so κ has matrix $K = (a_1 \ \ a_2 \ \ a_3)$, since then $\kappa(\mathbf{v}) = K\mathbf{v}$.

Further,

$$\lambda(\mathbf{e}_1) = \begin{pmatrix}0\\a_3\\-a_2\end{pmatrix}; \qquad \lambda(\mathbf{e}_2) = \begin{pmatrix}-a_3\\0\\a_1\end{pmatrix}; \qquad \lambda(\mathbf{e}_3) = \begin{pmatrix}a_2\\-a_1\\0\end{pmatrix};$$

so λ has matrix $L = \begin{pmatrix}0&-a_3&a_2\\a_3&0&-a_1\\-a_2&a_1&0\end{pmatrix}$, and then $\lambda(\mathbf{v}) = L\mathbf{v}$.

14. $V_{\leq k}$ is clearly a subspace: 0 is in $V_{\leq k}$; the sum of two elements in $V_{\leq k}$ is in $V_{\leq k}$, and any scalar multiple of something in $V_{\leq k}$ is in $V_{\leq k}$.

Then

$$\begin{aligned}i(1) &= x\\ i(x) &= \tfrac{1}{2}x^2\\ i(x^2) &= \tfrac{1}{3}x^3\\ i(x^3) &= \tfrac{1}{4}x^4\\ i(x^4) &= \tfrac{1}{5}x^5,\end{aligned}$$

so the matrix of i is $\begin{pmatrix}1&0&0&0&0\\0&\frac{1}{2}&0&0&0\\0&0&\frac{1}{3}&0&0\\0&0&0&\frac{1}{4}&0\\0&0&0&0&\frac{1}{5}\end{pmatrix}$.

15.

$$D(1) = 0, \qquad D(x) = 1, \qquad D(x^2) = 2x, \qquad D(x^3) = 3x^2,$$

so the matrix for D is $\begin{pmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\end{pmatrix}$. With the enlarged bases, we expect the matrix to be $\begin{pmatrix}0&1&0&0&0\\0&0&2&0&0\\0&0&0&3&0\\0&0&0&0&4\end{pmatrix}$.

16. We have

$$\begin{aligned} I(1) &= [x]_0^1 = 1 \\ I(x) &= [\tfrac{1}{2}x^2]_0^1 = \tfrac{1}{2} \\ I(x^2) &= [\tfrac{1}{3}x^3]_0^1 = \tfrac{1}{3} \\ I(x^3) &= [\tfrac{1}{4}x^4]_0^1 = \tfrac{1}{4} \\ I(x^4) &= [\tfrac{1}{5}x^5]_0^1 = \tfrac{1}{5}, \end{aligned}$$

so that the matrix is $(1 \ \ \tfrac{1}{2} \ \ \tfrac{1}{3} \ \ \tfrac{1}{4} \ \ \tfrac{1}{5})$.

17. (a) We have

$$\begin{aligned} \alpha(E_1) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \\ \alpha(E_2) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = E_2 - E_3 \\ \alpha(E_3) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = E_3 - E_2 \\ \alpha(E_4) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \end{aligned}$$

so the matrix is $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(b) We have

$$\begin{aligned} \beta(E_1) &= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = E_1 + E_3 \\ \beta(E_2) &= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = E_2 + E_4 \\ \beta(E_3) &= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = E_1 + E_3 \\ \beta(E_4) &= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = E_2 + E_4, \end{aligned}$$

so the matrix is $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

(c) We have

$$\begin{aligned} \alpha(\beta(E_1)) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -E_2 + E_3 \\ \alpha(\beta(E_2)) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = E_2 - E_3 \\ \alpha(\beta(E_3)) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -E_2 + E_3 \\ \alpha(\beta(E_4)) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = E_2 - E_3, \end{aligned}$$

so the matrix is $\begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(d) We have

$$AB = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = C.$$

18. Write $\mathbf{v}_1 = (1,1)^T$, $\mathbf{v}_2 = (1,-1)^T$. Then

$$\begin{aligned} A\mathbf{v}_1 &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\mathbf{v}_1 \\ A\mathbf{v}_2 &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0, \end{aligned}$$

so the matrix with respect to the new basis is $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$.

Alternatively, we can let $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ be the matrix formed from the columns of the new basis. Then $M^{-1}AM$ is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

19. Let $\mathbf{v}_1 = (0,1,-1)^T$, $\mathbf{v}_2 = (1,-1,1)^T$, $\mathbf{v}_3 = (-1,1,0)^T$. Then

$$\begin{aligned} A\mathbf{v}_1 &= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \mathbf{v}_1 \\ A\mathbf{v}_2 &= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \\ A\mathbf{v}_3 &= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -\mathbf{v}_3, \end{aligned}$$

so the matrix with respect to the new basis is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

20. (a) $\{e^{\lambda x}, xe^{\lambda x}, x^2e^{\lambda x}\}$.

(b) Write $f(x) = g(x)e^{\lambda x} \in V$. Then

$$f' = (g' + \lambda g)e^{\lambda x},$$

and if $g \in \mathbb{R}[x]_{\leq 2}$, then so is $g' + \lambda g$. Thus $f' \in V$.

(c) We have

$$\begin{aligned} D(e^{\lambda x}) &= \lambda e^{\lambda x} \\ D(xe^{\lambda x}) &= (1+\lambda x)e^{\lambda x} \\ D(x^2 e^{\lambda x}) &= (2x+\lambda x^2)e^{\lambda x}, \end{aligned}$$

so the matrix is $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 2 \\ 0 & 0 & \lambda \end{pmatrix}$.

(d) Then $D-\lambda = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. So $(D-\lambda)^2$ has matrix $\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $(D-\lambda)^3 = 0$. [We could also verify that $(D-\lambda)^3(f) = 0$ for each element in our basis.]

21. (a) ϕ is injective; the only element in the kernel is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. But it is not surjective; every vector in the image has its top and bottom co-ordinates identical. So $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, say, is not in the image.

(b) ϕ is not injective; the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a non-zero vector in the kernel. But it is surjective; given a vector $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$, it is the image of $\begin{pmatrix} a+b \\ b \\ 0 \end{pmatrix}$.

(c) This is injective; if $f(x) = ax^2 + bx + c$, then $\phi(f) = (c, b, 2a)^T$, so $\phi(f) = 0$ must mean that $a = b = c = 0$, so $f = 0$. It is also surjective; given any vector $(\lambda, \mu, \nu)^T$, we have $\phi(\frac{\nu}{2}x^2 + \mu x + \lambda) = (\lambda, \mu, \nu)^T$.

(d) This is not injective, as $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is in the kernel. It is also not surjective; the image can't possibly contain $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, say.

(e) This is not injective, as $f(x) = x$ is in the kernel. However, it is surjective: given $a \in \mathbb{R}$, the constant polynomial $f(x) = \frac{a}{2}$ maps to it.

22. (a) Given a sequence $a \in \ker(\pi)$, we have $a_0 = a_1 = 0$. Then $a_2 = 3a_1 - 2a_0 = 0$, and so on, so we see that $a_2 = a_3 = \cdots = 0$. (One could write this more formally by induction.)

(b) Put $u_0 = u_1 = 1$. Then $u_2 = 3 - 2 = 1$, $u_3 = 3 - 2 = 1$, etc., so $u_n = 1$ for all n lies in V. Similarly, $v_0 = 1$, $v_1 = 2$ gives $v_2 = 6 - 2 = 4$, etc. Let's prove that $v_n = 2^n$ lies in V. We have

$$3v_{n+1} - 2v_n = 3.2^{n+1} - 2.2^n = 6.2^n - 2.2^n = 4.2^n = 2^{n+2} = v_{n+2}.$$

(c) As $\pi(u) = (1,1)^T$ and $\pi(v) = (1,2)^T$, we see that $\pi(2u-v) = (2,2)^T - (1,2)^T = (1,0)^T$, and $\pi(-u+v) = (-1,-1)^T + (1,2)^T = (0,1)^T$, so we can put $b = 2u - v$ and $c = -u + v$.

(d) Suppose $v \in V$. Then $\pi(v) = (\alpha,\beta)^T$ for some $\alpha,\beta \in \mathbb{R}$. Then $\pi(v) = \alpha(1,0)^T + \beta(0,1)^T$. So $\pi(v) = \alpha\pi(b) + \beta\pi(c) = \pi(\alpha b + \beta c)$. Then $\pi(v - (\alpha b + \beta c))$ as π is linear. But π is injective, so $v - (\alpha b + \beta c) = 0$, and so $v = \alpha b + \beta c$.

(e) $\phi(u) = (1,1,\ldots) = u$, and $\phi(v) = (2,4,8,\ldots) = 2v$, so the matrix is $\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$.

23. If $(u,v)^T \in \ker(\phi)$, then $u = v = 0$, so ϕ is injective. The image is all matrices of the form $\begin{pmatrix}u & -u\\ -v & v\end{pmatrix}$, and all these matrices have the property that $A\begin{pmatrix}1\\1\end{pmatrix} = 0$. Conversely, given a matrix $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ such that $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$, then $a+b=0$ and $c+d=0$, so the matrix is of the form $\begin{pmatrix}a & -a\\ -d & d\end{pmatrix}$, as required.

24. Write $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$. Then notice that $\phi(A) = \begin{pmatrix}\frac{a-d}{2} & b\\ c & \frac{d-a}{2}\end{pmatrix}$. If $A \in \ker(\phi)$, then $\phi(A) = 0$ means that $a - d = b = c = 0$. So $a = d$, and $b = c = 0$. Then $A = \begin{pmatrix}a & 0\\ 0 & a\end{pmatrix} = aI$, as required. Also, we can see that the image consists of matrices whose trace is 0. Conversely, given a matrix of trace 0, $B = \begin{pmatrix}\alpha & \beta\\ \gamma & -\alpha\end{pmatrix}$, we can see that $\phi(B) = B$.

25. (a) If $f(x) = ax^2 + bx + c$, then $\phi(f) = (\frac{a}{3} - \frac{b}{2} + c, \frac{2a}{3} + 2c, \frac{a}{3} + \frac{b}{2} + c)^T$.

(b) If $f \in \ker(\phi)$, then $\frac{2a}{3} + 2c = 0$, so $a = -3c$. The other two equations then both reduce to $b = 0$. So f is of the form $-3cx^2 + c = c(1 - 3x^2)$.

(c) Set $g_+(x) = px + q$. Then $\phi(g_+) = (-\frac{p}{2} + q, 2q, \frac{p}{2} + q)^T$, so if we take $q = \frac{1}{2}$, and $p = -1$, then $\phi(-x + \frac{1}{2}) = (0,1,1)^T$.

(d) Then $g_-(x) = x + \frac{1}{2}$; $\phi(g_-) = (0,1,1)^T$.

(e) Certainly $\mathrm{im}(\phi) \subseteq \{(u,v,w)^T \mid v = u + w\}$, as the middle integral is the sum of the other two. However, since $(0,1,1)^T$ and $(1,1,0)^T$ are both in the image, and these are a basis for the given set, we conclude that the image is the whole of $\{(u,v,w)^T \mid v = u + w\}$.

26. (a) This map is linear, and $\phi(1-x) = (1,0)^T$ and $\phi(x) = (0,1)^T$ are in the image. So the image contains the span of $(1,0)^T$ and $(0,1)^T$, namely all of $\mathbb{R}^2$, and the map is surjective. The kernel consists of those f such that both 0 and 1 are roots, so that $x(x-1)|f$, i.e., f should be divisible by $x^2 - x$. Since f has degree at most 3, $f(x) = (ax+b)(x^2-x) = a(x^3-x^2) + b(x^2-x)$, so $x^3 - x^2$ and $x^2 - x$ are a basis for the kernel.

(b) If f is in the kernel, then f is a quadratic with roots at 0, 1, 2 and 3; however, the only quadratic with more than 2 roots is the zero polynomial. We also have

$$\psi(ax^2 + bx + c) = (c, a+b+c, 4a+2b+c, 9a+6b+c)^T,$$

and one can do row operations to see that the only relation is the given one. Another way to see this is to use the rank-nullity theorem; the kernel is trivial, so the dimension of the image should be the same as the dimension of $\mathbb{R}[x]_{\leq 2}$, namely 3. Since the image is inside the space V (which is easily checked), and V is 3-dimensional, the image must be all of V.

To find bases so that the matrix of ψ has the required form, we start by choosing a basis for the kernel. But ψ is injective, so this is empty. We then extend it to a basis for all of $\mathbb{R}[x]_{\leq 2}$, by choosing $\mathcal{V} = 1, x, x^2$, say (other choices are possible). Then consider $\psi(1), \psi(x), \psi(x^2)$, which are $(1,1,1,1)^T, (0,1,2,3)^T, (0,1,4,9)^T$ respectively, and we extend this set to a basis for $\mathbb{R}^4$ by adding, say, $(1,0,0,0)^T$, which is not in the image of ψ as it is not in V. We may therefore use the bases $v_1 = 1$, $v_2 = x$, $v_3 = x^2$ for $\mathbb{R}[x]_{\leq 2}$ and $w_1 = (1,1,1,1)^T$, $w_2 = (0,1,2,3)^T$, $w_3 = (0,1,4,9)^T$ and $w_4 = (1,0,0,0)^T$ for $\mathbb{R}^4$; then the matrix has the desired form.

27. Explicitly, we have

$$\phi(ax^2 + bx + c) = \begin{pmatrix} -a+c & b \\ -b & -a+c \end{pmatrix}.$$

So $ax^2 + bx + c$ is in the kernel if $-a + c = b = 0$, i.e., $f(x) = a(x^2 + 1)$. So a basis of $\ker(\phi)$ is $x^2 + 1$. A matrix in the image is necessarily

$$(-a+c)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so a basis for $\mathrm{im}(\phi)$ is $\{I, J\}$.

28. If $\ker(\phi) = 0$, then ϕ is injective; by the rank-nullity theorem, its image would then have dimension 2, so the map would also be surjective, and ϕ will be an isomorphism.

We know that $(x, y) \in \ker(\phi)$ if $ax + by = 0$ and $cx + dy = 0$. Multiply the first equation by c and the second by b, and subtract to eliminate y, we get $(ad-bc)x = 0$. If $ad - bc \neq 0$, then x must be 0, and substituting back gives $y = 0$ also. So in this case, $x = y = 0$, and the only element in the kernel is $(0, 0)$. Thus in this case ϕ is an isomorphism.

Conversely, if $ad - bc = 0$, the element $(-d, c) \in \ker(\phi)$, since $\phi(-d, c) = (-ad + bc, -cd + dc) = (0, 0)$.

29. If $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then a calculation gives

$$\begin{aligned} \phi(A) &= (1 \;\; x \;\; x^2) A \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} \\ &= a + (b+d)x + (c+e+g)x^2 + (f+h)x^3 + ix^4. \end{aligned}$$

We can now see that ϕ is surjective; given a polynomial $\alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \epsilon$, we can use the matrix $A = \begin{pmatrix} \epsilon & \delta & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & \alpha \end{pmatrix}$, for example.

The kernel consists of all matrices with $a = b + d = c + e + g = f + h = i = 0$. In other words, $d = -b$, $g = -c - e$ and $h = -f$, and the general form of a matrix in the kernel is $\begin{pmatrix} 0 & b & c \\ -b & e & f \\ -c-e & -f & 0 \end{pmatrix}$ which is

$$b\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + e\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

and so the four matrices appearing in this expression form a basis for $\ker(\phi)$.

30. Let $w_1, w_2 \in W$. As ϕ is an isomorphism, it is certainly surjective, so $w_1 = \phi(v_1)$ and $w_2 = \phi(v_2)$ for some v_1, $v_2 \in V$. By linearity, $\phi(v_1 + v_2) = w_1 + w_2$. Then

$$\phi^{-1}(w_1 + w_2) = v_1 + v_2 = \phi^{-1}(w_1) + \phi^{-1}(w_2).$$

Similarly, if $w \in W$, then $w = \phi(v)$ for some $v \in V$, and linearity gives $\alpha w = \phi(\alpha v)$. Then

$$\phi^{-1}(\alpha w) = \alpha v = \alpha\phi^{-1}(w),$$

and it follows that ϕ^{-1} is linear.

31. Since any non-constant polynomial has non-zero derivative, and constants have zero derivative, the kernel is simply the set of all constant polynomials. Since the kernel has dimension 1, the image must have dimension n (recall that $\mathbb{R}[x]_{\leq n}$ has dimension $n+1$). Since taking the derivative of a polynomial reduces its degree by 1, the image is contained in the set of polynomials of degree at most $n-1$. Since the image is an n-dimensional subspace of this n-dimensional space $\mathbb{R}[x]_{\leq n-1}$, we must have equality. (Of course one can also do this explicitly, by showing that any polynomial of degree at most $n-1$ is the derivative of a polynomial of degree at most n.)

32. One example is $\phi(f) = \frac{d^2 f}{dx^2}$. Another is $\theta(ax^3 + bx^2 + cx + d) = ax + d$ (or any other expression involving two monomials).

33. (a) The image consists of all vectors $(t, t, t)^T$, so has rank 1 and nullity 2 (and the kernel is all $(x, y, z)^T$ with $x + y + z = 0$). Indeed, it suffices to work out the ranks of the given matrices, as in MAS201.
    (b) The matrix has rank 3, and nullity 0.
    (c) The matrix has rank 0, and nullity 3.
    (d) The matrix has rank 3, and nullity 0.
    (e) The matrix has rank 2, and nullity 1.

34. Suppose that V has dimension n. Then (a) says that $\ker(\phi) = 0$, so has dimension 0. By the rank-nullity theorem, this is equivalent to the image of ϕ having dimension n, i.e., ϕ is surjective. So (a) and (b) are equivalent. By definition of isomorphism, (c) is equivalent to both (a) and (b). It remains to show that (d) is equivalent to any of the first three. But if ϕ is an isomorphism, it is invertible, and so the same will be true of the matrix representing it. This is equivalent to the non-vanishing of the determinant of this matrix A.

    (Incidentally, the result is false if V does not have finite dimension; for example, the map $D : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ given by $D(f) = \frac{df}{dx}$ is surjective, but not injective, as constant polynomials are in the kernel.)

35. For this, we need to explain how we add two maps, and how to scale a map by a real number. We do the obvious thing; given two maps ϕ and θ, say, we define their sum $\phi+\theta$ to be the map whose effect on a vector v is to send it to the sum of $\phi(v)$ and $\theta(v)$. That is,

$$(\phi+\theta)(v) = \phi(v) + \theta(v).$$

Similarly, if α is a real number,

$$(\alpha\phi)(v) = \alpha(\phi(v)).$$

Let's check that the first of these is linear (the second is similar, but easier).

$$\begin{aligned}
(\phi+\theta)(\alpha v + \alpha' v') &= \phi(\alpha v + \alpha' v') + \theta(\alpha v + \alpha' v') \\
&\qquad \text{(by definition of the sum)} \\
&= \alpha\phi(v) + \alpha'\phi(v') + \alpha\theta(v) + \alpha'\theta(v') \\
&\qquad \text{(by linearity of } \phi \text{ and } \theta\text{)} \\
&= \alpha(\phi(v)+\theta(v)) + \alpha'(\phi(v')+\theta(v')) \\
&= \alpha(\phi+\theta)(v) + \alpha'(\phi+\theta)(v') \\
&\qquad \text{(by definition of the sum again)}
\end{aligned}$$

as required. Also, $L(V,W)$ is non-empty; the zero map (which maps everything in V to 0_W) is going to be the zero element of $L(V,W)$. One can check all the usual rules hold.

It follows that $L(V,W)$ is a vector space.

We know that if V is isomorphic to $\mathbb{R}^m$ and W to $\mathbb{R}^n$, then linear maps are given by $n\times m$ matrices (once bases are chosen). Since there are nm matrices of the right size, we find that $\dim L(V,W) = nm$.

36. (a) The natural choice is

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

We have

$$\begin{aligned}
\gamma(E_1) &= TE_1 - E_1T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = E_3 - E_2 \\
\gamma(E_2) &= TE_2 - E_2T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = E_4 - E_1 \\
\gamma(E_3) &= TE_3 - E_3T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = E_1 - E_4 \\
\gamma(E_4) &= TE_4 - E_4T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = E_2 - E_3,
\end{aligned}$$

so γ has matrix $\begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$.

(b) The kernel contains E_1+E_4 and E_2+E_3; the image contains E_1-E_4 and E_2-E_3, so $\mathrm{Sp}(E_1+E_4, E_2+E_3) \subseteq \ker(\gamma)$ and $\mathrm{Sp}(E_1-E_4, E_2-E_3) \subseteq \mathrm{im}(\gamma)$. As the dimensions must add to 4, by the rank-nullity theorem, these inclusions must be equalities, and the given matrices are bases. Since $\langle E_1+E_4, E_1-E_4\rangle = 0$ and $\langle E_1+E_4, E_2-E_3\rangle = 0$, E_1+E_4 is orthogonal to $\ker(\gamma)$, and similarly E_2+E_3 is orthogonal to $\ker(\gamma)$. Thus $\mathrm{im}(\gamma)$ is orthogonal to $\ker(\gamma)$; this shows that $\mathrm{im}(\gamma) \subseteq (\ker(\gamma))^\perp$, but both must be 2-dimensional, and we have equality.

(c) $\gamma^2 = \begin{pmatrix} 2 & 0 & 0 & -2 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix}$, and so $\gamma^4 = \begin{pmatrix} 8 & 0 & 0 & -8 \\ 0 & 8 & -8 & 0 \\ 0 & -8 & 8 & 0 \\ -8 & 0 & 0 & 8 \end{pmatrix} = 4\gamma^2$, if we use the given basis.

(d) E_1+E_4 and E_2+E_3 both lie in the kernel, so are eigenvectors with eigenvalue 0. The other eigenvalues of γ are ± 2; we can find the eigenvectors explicitly in the usual way, and we find that $E_1 - E_2 + E_3 - E_4$ is an eigenvector with eigenvalue 2, and $E_1 + E_2 - E_3 - E_4$ is an eigenvector with eigenvalue -2.