MAS277 2013-2014 SAMPLE EXAM SOLUTIONS

Question 1:

(i) (a) A set $U \subseteq V$ is called a *subspace* of V if the following conditions hold:

- $\mathbf{0} \in U$.
- $\mathbf{u} + \mathbf{v} \in U$ whenever $\mathbf{u}, \mathbf{v} \in U$.
- $c\mathbf{v} \in U$ for all $\mathbf{v} \in U$ and $c \in \mathbb{F}$.

(b) The span of U is defined by

$$\mathrm{sp}(U) := \{\mathbf{u} \in V : \text{ there exists } \mathbf{u}_1, \ldots \mathbf{u}_k \in U, c_1, \ldots, c_k \in \mathbb{F} \text{ with } \mathbf{u} = c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k\}.$$

The set U is said to span W if $\mathrm{sp}(U) = W$.

(c) If U and W are subspaces of V then V is said to be the direct sum of U and W if $U \cap W = \{\mathbf{0}\}$ and $U + W = V$ where

$$U + W := \{\mathbf{u} + \mathbf{w} : \mathbf{u} \in U, \mathbf{w} \in W\}.$$

(ii) (a) The set $U + W$ is defined in Question 1 (c). To show that $U + W$ is a subspace we need to show that

- $\mathbf{0} \in U + W$
- $\mathbf{u} + c\mathbf{v} \in U + W$ whenever $\mathbf{u}, \mathbf{v} \in U + W$ and $c \in \mathbb{F}$.

Firstly, we know $\mathbf{0} \in U, W$ as both U and W are subspaces. So $\mathbf{0} = \mathbf{0} + \mathbf{0}$ is a sum of elements in U and W. Namely, $\mathbf{0} \in U + W$. Secondly, if $\mathbf{x}, \mathbf{y} \in U + W$ then there exists $\mathbf{u}_1, \mathbf{u}_2 \in U$ and $\mathbf{w}_1, \mathbf{w}_2 \in W$ such that $\mathbf{x} = \mathbf{u}_1 + \mathbf{w}_1$ and $\mathbf{y} = \mathbf{u}_2 + \mathbf{w}_2$. If $c \in \mathbb{F}$ then $\mathbf{u}_1 + c\mathbf{u}_2 \in U$ and $\mathbf{w}_1 + c\mathbf{w}_2 \in W$ because U and W are subspaces of V. So

$$\mathbf{x} + c\mathbf{y} = (\mathbf{u}_1 + \mathbf{w}_1) + c(\mathbf{u}_2 + \mathbf{w}_2) = (\mathbf{u}_1 + c\mathbf{u}_2) + (\mathbf{w}_1 + c\mathbf{w}_2) \in U + W.$$

Hence $U + W$ is a subspace of V.

(b) We know that $U+W$ is a subspace of $\mathbb{R}^4$ so $\dim(U+W) \leq \dim(\mathbb{R}^4) = 4$. We consider the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in U$ and $\mathbf{w} \in W$ given by

$$\mathbf{u}_1 := \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 := \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{u}_3 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{w} := \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

We will show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{w}\}$ is a linearly independent set. If $a, b, c, d \in \mathbb{F}$ and $a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3 + d\mathbf{w} = \mathbf{0}$ then, by comparing entries of the vectors, we find

$$a + b + c + d = d - a = d - b = d - c = 0.$$

It is easy to see that $a = b = c = d = 0$. This means that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{w}\} \subseteq U+W$ is a linearly independent set. Hence $4 \leq \dim(U+W) \leq \dim(\mathbb{R}^4) = 4$ and so $\dim(U+W) = 4$.

(iii) We have $S \subseteq M_{3\times 3}(\mathbb{R})$. So S is a vector space if and only if S is a subspace of $M_{3\times 3}(\mathbb{R})$. We know that S is a subspace of $M_{3\times 3}(\mathbb{R})$ if and only if $\mathbf{0} \in S$ and $A + cB \in S$ whenever $A, B \in S$ and $c \in \mathbb{R}$. The zero vector $\mathbf{0}$ in $M_{3\times 3}(\mathbb{R})$ is the zero matrix O. Clearly $O = O^T$ so $\mathbf{0} \in S$. If $A = (a_{ij}), B = (b_{ij}) \in S$ then $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$ for all i, j. So if $c \in \mathbb{R}$ then the ij^{th} entry of $D = A + cB$ is $d_{ij} = a_{ij} + cb_{ij} = a_{ji} + cb_{ji} = d_{ji}$. Namely $A + cB \in S$ and hence S is a vector space.

(iv) Let $\dim(V) = n$. Define $U_1 := \{\mathbf{u}_1\}$ for some non-zero vector $\mathbf{u}_1 \in S$. If $\mathrm{sp}(U_1) = V$ then $\{\mathbf{u}_1\} \subseteq S$ is a basis for V. Now define U_k recursively by

$$\mathrm{sp}(U_{k+1}) := \begin{cases} U_k & \text{if } \mathrm{sp}(U_k) = V; \\ U_k \cup \{\mathbf{u}_{i_{k+1}}\} & \text{for some } \mathbf{u}_{i_{k+1}} \in S \backslash \mathrm{sp}(U_k). \end{cases}$$

Each U_k is linearly independent and contained in S by construction. So U_n is a linearly independent set with n elements. Hence $U_n \subseteq S$ is a basis for V.

Question 2:

(i) (a) $L : V \to W$ is said to be a *linear map* if for all $u, v \in V$ and $c \in \mathbb{F}$ we have

- $L(\mathbf{u} + \mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$
- $L(c\mathbf{v}) = cL(\mathbf{v})$.

(b) Let $I_V : V \to V$ and $I_W : W \to W$ denote the identity maps on the vector spaces V and W respectively. A linear map $L : V \to W$ between vector spaces is said to be *invertible* if there exists a map $L' : W \to V$ such that $L \circ L' = I_W$ and $L' \circ L = I_V$.

(c) Two vector spaces V and W are said to be *isomorphic* if there exists an invertible linear map $L : V \to W$.

(ii) (a) The change of coordinate matrix is given by

$$[I]_B^C := \begin{pmatrix} | & | & | & | \\ [I(1)]_C & [I(x)]_C & [I(x^2)]_C & [I(x^3)]_C \\ | & | & | & | \end{pmatrix}$$

where $I : \mathbb{R}[x]_{\leq 3} \to \mathbb{R}[x]_{\leq 3}$ is the identity transformation. As I is the identity map we have $I(1) = 1$, $I(x) = x$, $I(x^2) = x^2$, $I(x^3) = x^3$. Furthermore

$$\begin{aligned}
1 &= \tfrac{1}{2}(1 + x + x^3) - \tfrac{1}{2}(x) + \tfrac{1}{2}(1 - x^3) + 0(1 + x + x^2 + x^3) \\
x &= 0(1 + x + x^3) + 1(x) + 0(1 - x^3) + 0(1 + x + x^2 + x^3) \\
x^2 &= -1(1 + x + x^3) + 0(x) + 0(1 - x^3) + 1(1 + x + x^2 + x^3) \\
x^3 &= \tfrac{1}{2}(1 + x + x^3) - \tfrac{1}{2}(x) + \tfrac{1}{2}(1 - x^3) + 0(1 + x + x^2 + x^3)
\end{aligned}$$

Therefore the change of basis matrix is given by

$$[I]_B^C = \begin{pmatrix} \frac{1}{2} & 0 & -1 & \frac{1}{2} \\ -\frac{1}{2} & 1 & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(b) We know that $[I]_B^C[p(x)]_B = [p(x)]_C$. Hence

$$[p(x)]_C = \begin{pmatrix} \frac{1}{2} & 0 & -1 & \frac{1}{2} \\ -\frac{1}{2} & 1 & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \frac{1}{2}a - c + \frac{1}{2}d \\ -\frac{1}{2}a + b - \frac{1}{2}d \\ \frac{1}{2}a - \frac{1}{2}d \\ c \end{pmatrix}.$$

(iii) If $\det A \neq 0$ then $A\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$. Hence $\ker(\phi_A) = \{\mathbf{0}\}$ so $\dim(\ker(\phi_A)) = 0$. The rank-nullity theorem tells us that

$$4 = \dim(M_{2\times 2}(\mathbb{R})) = \dim(\mathrm{Im}(\phi_A)) + \ker(\phi_A) = \dim(\mathrm{Im}(\phi_A)) + 0.$$

Hence $\dim(\mathrm{Im}(\phi_A)) = 4$.

(iv) The trace of a linear operator $L : V \to V$ on a finite dimensional vector space is defined to be $[\mathrm{trace}(L)]_B^B$ for any basis B of V. In our case $V = \mathbb{R}[x]_{\leq n}$ and we take $B = \{1, x, \dots, x^n\}$. So, for $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, we have

$$L(p(x)) = x(a_1 + 2a_2 + \cdots + na_nx^{n-1}) + a_0 = a_0 + a_1x + 2a_2x^2 + \cdots + na_nx^n.$$

So $\mathrm{trace}(L)]_B^B$ is the $(n+1) \times (n+1)$ diagonal matrix whose ii^{th} diagonal entry is $(i-1)$ if $i \neq 1$ and 1 if $i = 1$. Namely

$$[(L)]_B^B = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n \end{pmatrix}.$$

Hence, $\mathrm{trace}(L) = 1 + 1 + 2 + \cdots + n = 1 + \frac{n(n+1)}{2}$.

Question 3:

(i) (a) $\langle , \rangle : V \times V \to \mathbb{R}$ is a real inner product if $(\mathbf{u}, \mathbf{v}) \mapsto \langle \mathbf{u}, \mathbf{v} \rangle \in \mathbb{R}$ satisfies the following properties:

- $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.
- $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$ for all $\mathbf{u}, \mathbf{v} \in V$ and $c \in \mathbb{R}$.
- $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ for all $\mathbf{u}, \mathbf{v} \in V$.
- We have $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ for all $\mathbf{v} \in V \backslash \{\mathbf{0}\}$.

(ii) (a) The Cauchy-Schwarz inequality states: if V is an inner product space and $\mathbf{u}, \mathbf{v} \in V$ then the following inequality holds:

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\| .$$

Furthermore, $|\langle \mathbf{u}, \mathbf{v} \rangle| = \|\mathbf{u}\| \|\mathbf{v}\|$ if and only if $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$.

(b) We use $\|\mathbf{x}\|^2 := \langle \mathbf{x}, \mathbf{x} \rangle$ and the linearity properties of the inner product to see that

$$\begin{aligned} \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle &= \langle \mathbf{u}, \mathbf{u} \rangle + 2\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle = \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2 \\ &\leq \|\mathbf{u}\|^2 + 2|\langle \mathbf{u}, \mathbf{v} \rangle| + 2\|\mathbf{v}\|^2 \leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\| \|\mathbf{v}\| + 2\|\mathbf{v}\|^2 \end{aligned}$$

where the second inequality is obtained using the Cauchy-Schwarz inequality.

(c) When answering part (b) we saw that

$$\langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle = \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2 .$$

Hence $\langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle = \|\mathbf{u}\|^2 + 2\|\mathbf{u}\| \|\mathbf{v}\| + \|\mathbf{v}\|^2$ if and only if $\langle \mathbf{u}, \mathbf{v} \rangle = \|\mathbf{u}\| \|\mathbf{v}\|$.

(iii) The triangle inequality states that if V be an inner product space then for $\mathbf{u}, \mathbf{v} \in V$ the following inequality holds:

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\| .$$

For the inner product space $C[0,1]$ of continuous functions on $[0,1]$ with the usual inner product defined by

$$\langle f(x), g(x)\rangle := \int_0^1 f(x)g(x)\mathrm{dx}$$

the triangle inequality tells us that

$$\|f(x)+g(x)\| = \sqrt{\int_0^1 (f(x)+g(x))^2\mathrm{dx}} \leq \sqrt{\int_0^1 (f(x))^2\mathrm{dx}} + \sqrt{\int_0^1 (g(x))^2\mathrm{dx}}$$
$$= \|f(x)\| + \|g(x)\|\,.$$

Setting $f(x) = x^{60}$ and $g(x) = e^x$ we find that

$$\sqrt{\int_0^1 (x^{60}+e^x)^2\mathrm{dx}} \leq \sqrt{\int_0^1 (x^{60})^2\mathrm{dx}} + \sqrt{\int_0^1 (e^x)^2\mathrm{dx}}$$
$$= \sqrt{\int_0^1 x^{120}\mathrm{dx}} + \sqrt{\int_0^1 e^{2x}\mathrm{dx}} = \frac{1}{11} + \sqrt{\frac{e^2-1}{2}}.$$

(iv) We know that the adjoint is unique. Using the standard rules for an inner product and the definition of the adjoint (namely $\langle L(\mathbf{u}), \mathbf{v}\rangle = \langle \mathbf{u}, L^*(\mathbf{v})\rangle$) we see that

$$\langle LT(\mathbf{u}), \mathbf{v}\rangle = \langle L(T(\mathbf{u})), \mathbf{v}\rangle = \langle T(\mathbf{u}), L^*\mathbf{v}\rangle = \langle \mathbf{u}, T^*(L^*(\mathbf{v}))\rangle = \langle \mathbf{u}, T^*L^*(\mathbf{v})\rangle.$$

Hence $(LT)^* = T^*L^*$.

Question 4:

(i) (a) $U \subseteq V$ is said to be an orthonormal basis for V if

- U is a basis for V
- $\|\mathbf{u}\| = 1$ for all $\mathbf{u} \in U$
- if $\mathbf{u}, \mathbf{v} \in U$ with $\mathbf{u} \neq \mathbf{v}$ then $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

(b)

$$U^{\perp} := \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{u} \rangle = 0 \text{ for all } \mathbf{u} \in U\}.$$

(ii) The Gram-Schmidt method of orthogonalisation tells us that if $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a linearly independent set of vectors then $C = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is an orthogonal set with $\mathrm{sp}(C) = \mathrm{B}$ where

$$\mathbf{u}_1 := \mathbf{v}_1, \qquad \text{and} \qquad \mathbf{u}_i := \mathbf{v}_i - \sum_{j=1}^{i-1} \frac{\langle \mathbf{v}_i, \mathbf{u}_j \rangle}{\langle \mathbf{u}_j, \mathbf{u}_j \rangle} \mathbf{u}_j \qquad \text{for } i \geq 2.$$

So, $\mathbf{u}_1 = \mathbf{v}_1$, $\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\mathbf{u}_1$, $\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle}\mathbf{u}_2$. In our case $\mathbf{v}_1 = (1, 1, 1, 0)^T, \mathbf{v}_2 = (1, -1, 0, 1)^T, \mathbf{v}_3 = (0, 1, 1, 1)^T$, and $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 0$. It is now easy to see that $\mathbf{u}_1 = \mathbf{v}_1$, $\mathbf{u}_2 = \mathbf{v}_2$, $\mathbf{u}_3 = (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 1)$. Hence

$$B' = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix} \right\}$$

is an orthogonal basis for U.

(iii) We know that the distance from a subspace to a vector is minimised at the projection. In our case this means $\mathbf{x} = \mathrm{Proj}_U((1, 1, 1, 1)^T)$. The projection $\mathrm{Proj}_U(\mathbf{v})$ of a vector $\mathbf{v}$ in a vector space V onto a subspace U with an orthogonal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is given by

$$\mathrm{Proj}_U(\mathbf{v}) = \sum_{i=1}^{k} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i.$$

Taking the $\mathbf{u}_i$ from part (ii) and $\mathbf{v} = (1,1,1,1)^T$ we get

$$\mathrm{Proj}_U(\mathbf{v}) = \frac{3}{3}\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\-1\\0\\1\end{pmatrix} + \frac{3}{5}\begin{pmatrix}-\frac{2}{3}\\\frac{1}{3}\\\frac{1}{3}\\1\end{pmatrix} = \begin{pmatrix}\frac{14}{15}\\\frac{13}{15}\\\frac{18}{15}\\\frac{14}{15}\end{pmatrix}.$$

We can now easily see $\|\mathbf{v} - \mathrm{Proj}_U(\mathbf{v})\|^2 = (\frac{1}{15})^2 + (\frac{2}{15})^2 + (-\frac{3}{15})^2 + (\frac{1}{15})^2 = \frac{1}{15}$. Hence

$$\min\left\{\left\|(1,1,1,1)^T - \mathbf{x}\right\| \ : \ \mathbf{x} \in U\right\} = \frac{1}{\sqrt{15}}.$$

(iv) Let $C^{2\pi}$ denote the inner product space of 2π periodic functions on $\mathbb{R}$ with inner product defined by

$$\langle f(x), g(x)\rangle := \int_{-\pi}^{\pi} f(x)g(x)\mathrm{dx}.$$

Define T_n to be the subspace of $C^{2\pi}$ given by

$$T_n := \mathrm{sp}\{1, \sin(x), \cos(x), \dots, \sin(nx), \cos(nx)\}.$$

The convergence of Fourier series theorem tells us that for any $f(x) \in C^{2\pi}$ we have

$$\left\|f(x) - \mathrm{Proj}_{T_n}(f(x))\right\| \to 0 \qquad \text{as} \qquad n \to \infty.$$