The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS

Vector spaces and Fourier theory SAMPLE EXAM **2 hours**

Attempt all of the questions. Each of the four questions is worth 25 marks, with the specific allocation shown in brackets.

1 Throughout this question, V will denote a vector space over a field $\mathbb{F}$.

(i) Let U and W be sets contained in V. Define what it means for

(a) U to be a subspace of V;

(b) U to span W;

(c) If U and W are subspaces of V what does it mean to say that V is the direct sum of U and W. ***(8 marks)***

(ii) (a) Show that if U and W are subspaces of V then $U+W$ is a subspace of V.

(b) Let U, W be the subspaces of $\mathbb{R}^4$ given by

$$U := \{(x,y,z,w) \in \mathbb{R}^4 \ : \ x+y+z+w=0\}$$
$$W := \{(x,y,z,w) \in \mathbb{R}^4 \ : \ x-y+z-w=0\}.$$

Find $\dim(U+W)$. ***(8 marks)***

(iii) Is $S = \{A \in M_{3\times 3}(\mathbb{R}) \,|\, A = A^T\}$ a vector space? You should justify your answer by providing a proof or by demonstrating that one of the vector space axioms fails. ***(5 marks)***

(iv) Suppose that $S \subseteq V$ is a set with infinitely many distinct elements and $\mathrm{sp}(S)$ is finite dimensional. Show that S contains a basis for $\mathrm{sp}(S)$. ***(4 marks)***

2 (i) Let V and W be vector spaces over a field $\mathbb{F}$. Let $L : V \to W$ be a map.

(a) Define what it means for L to be linear.

(b) Define what it means for L to be invertible.

(c) Define what it means for V and W to be isomorphic. ***(8 marks)***

(ii) Let $B, C \subseteq \mathbb{R}[x]_{\leq 3}$ be the ordered bases for $\mathbb{R}[x]_{\leq 3}$ defined by

$$B := \{1, x, x^2, x^3\}, \qquad C := \{1 + x + x^3, x, 1 - x^3, 1 + x + x^2 + x^3\}.$$

(a) Calculate the change of basis matrix $[I]_B^C$.

(b) Calculate the coordinate vector $[p(x)]_C$ where $p(x)$ is the polynomial

$$p(x) := a + bx + cx^2 + dx^3$$

(8 marks)

(iii) Let $\phi_A : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear operator defined by $B \mapsto AB$ for some $A \in M_{2\times 2}(\mathbb{R})$ with $\det(A) \neq 0$. Find $\dim(\mathrm{Im}(L))$. ***(5 marks)***

(iv) Let $L : \mathbb{R}[x]_{\leq n} \to \mathbb{R}[x]_{\leq n}$ be the linear operator defined by

$$p(x) \mapsto xp'(x) + p(0).$$

Express $\mathrm{trace}(L)$ in terms of n. ***(4 marks)***

3 (i) Let V be a vector space over $\mathbb{R}$. Define what it means to say that $\langle , \rangle : V \times V \to \mathbb{R}$ is a real inner product on V. ***(8 marks)***

(ii) Let V be a real inner product space.

(a) State the Cauchy-Schwarz inequality and show that for all $\mathbf{u}, \mathbf{v} \in V$ the following inequality holds

$$\langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle \leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2.$$

(b) State a necessary and sufficient condition on $\mathbf{u}$ and $\mathbf{v}$ for equality to hold in part (ii). ***(8 marks)***

(iii) State the triangle inequality and show that

$$\sqrt{\int_0^1 (x^{60} + e^x)^2 \mathrm{dx}} \leq \frac{1}{11} + \sqrt{\frac{e^2 - 1}{2}}.$$

(5 marks)

(iv) Let L and T be linear operators on an inner product space V with an adjoints L^* and T^* respectively. Show that $(LT)^* = T^* L^*$. ***(4 marks)***

4 (i) Let V be a real inner product space and let $U \subset V$.

(a) Define what it means to say $U \subseteq V$ is an orthonormal basis.

(b) Define $U^{\perp}$. ***(8 marks)***

(ii) Consider the real inner product space $\mathbb{R}^4$, with inner product $\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x} \cdot \mathbf{y}$ defined by the usual dot product on $\mathbb{R}^4$. Let U be the subspace of $\mathbb{R}^4$ with basis

$$B := \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \right\}$$

By applying the Gram-Schmidt orthogonalisation process to U find an orthogonal basis for U. ***(8 marks)***

(iii) For U in part (ii), find

$$\min \left\{ \left\| (1,1,1,1)^T - \mathbf{x} \right\| \; : \; \mathbf{x} \in U \right\}.$$

(5 marks)

(iv) State the convergence of Fourier series theorem for continuous 2π periodic functions. ***(4 marks)***

End of Question Paper