The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012–2013**

Vector spaces and Fourier theory **2 hours**

Attempt all of the questions. Each of the four questions is worth 25 marks, with the specific allocation shown in brackets.

1 (i) Define what it means for a vector space V to be *finite-dimensional.*
(2 marks)

(ii) Define the *dimension* of a finite-dimensional vector space V, and explain (without proof of details) why every finite-dimensional vector space has a dimension. ***(6 marks)***

(iii) Let U, W be subspaces of a finite-dimensional vector space V.

(a) Using only your definition above and the Steinitz Exchange Lemma, show that U (and hence W) is also finite-dimensional. ***(4 marks)***

(b) Assume that $\dim U + \dim W = \dim V$. Show that $V = U \oplus W$ if and only if $U \cap W = 0$. ***(5 marks)***

(c) Let $V = M_3(\mathbb{R})$. Let U be the subspace of anti-symmetric matrices (i.e., matrices A such that $A^T = -A$) and let W be the subspace of upper-triangular matrices (i.e., matrices $A = [a_{ij}]$ such that $a_{ij} = 0$ if $i > j$). Use the previous result to prove that $V = U \oplus W$.
(8 marks)

2 Let V, W be finite-dimensional vector spaces.

(i) Let $F : V \to W$ be a linear transformation. Explain how to represent F using a matrix by choosing a basis $\mathcal{V}$ for V and a basis $\mathcal{W}$ for W.

(5 marks)

(ii) Let $[F_{\mathcal{V},\mathcal{W}}]$ denote the matrix associated to F by the above process. If $F : V \to V$ is a linear transformation, and $\mathcal{V}, \mathcal{W}$ are two different bases for V, state the relationship between the matrices $[F_{\mathcal{V},\mathcal{V}}]$ and $[F_{\mathcal{W},\mathcal{W}}]$. Use this result to show that the determinant

$$\det[F_{\mathcal{V},\mathcal{V}}]$$

of a linear transformation $F : V \to V$ is independent of the choice of basis, i.e., that

$$\det[F_{\mathcal{V},\mathcal{V}}] = \det[F_{\mathcal{W},\mathcal{W}}].$$

(5 marks)

(iii) Define the *kernel* and *image* of a linear transformation $F : V \to W$. Define what it means for F to be an *isomorphism*. Explain (without proof of details) the relationship between F being an isomorphism, and properties of its kernel and image. ***(7 marks)***

(iv) Recall that a trigonometric polynomial of degree n is a sum of the form

$$\frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt)$$

with $a_i, b_i \in \mathbb{R}$. Let V be the vector space of trigonometric polynomials of degree less than or equal to 2, and let $E : V \to V$ be given by sending a trigonometric polynomial $f(x)$ to

$$f''(x) - 2f'(x) + f(x).$$

(a) Verify that E is a linear transformation. ***(3 marks)***

(b) Find the matrix $[E_{\mathcal{V},\mathcal{V}}]$ where $\mathcal{V}$ is the basis $1, \sin x, \cos x, \sin 2x, \cos 2x$, and thus compute $\det E$. ***(5 marks)***

3 Recall that $C^\infty([0,1])$ is the vector space of smooth functions $[0,1] \to \mathbb{R}$, i.e., the vector space of functions $[0,1] \to \mathbb{R}$ which can be differentiated as many times as desired.

(i) Let $F : V \to W$ be a linear transformation between inner product spaces. Define the *adjoint* of F (assuming it exists). ***(3 marks)***

(ii) Let $f \in C^\infty([0,1])$; recall that $f^{(n)}$ is notation for the nth derivative of f, where $f^{(0)}$ is just the function f. Define

$$V = \{f \in C^\infty([0,1]) : f^{(n)}(0) = 0 = f^{(n)}(1) \text{ for all } n \geq 0\}.$$

Show that V is a subspace of $C^\infty([0,1])$, and that the function $D : V \to V$ given by

$$D(f) = f'$$

is a linear transformation. ***(7 marks)***

(iii) Using the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)$$

on V, find the adjoint D^* of D. ***(8 marks)***

(iv) Let $D^2 : V \to V$ be defined by $D^2(f) = f''$. Show that D^2 is self-adjoint. ***(7 marks)***

4 Recall that the standard basis $\{e_i\}$ of $\mathbb{R}^n$ has elements e_i where the ith coordinate of e_i is 1 but every other coordinate is 0.

(i) Show that the matrix

$$A = \begin{bmatrix} 1 & -4 \\ -4 & 20 \end{bmatrix}$$

gives an inner product on $\mathbb{R}^2$ using the formula

$$\langle x, y \rangle_A = x^T A y.$$

(Hint: to show that $\langle v, v \rangle_A \geq 0$, try writing the resulting number as a sum of squares.) ***(8 marks)***

(ii) Define the *norm* of a vector in a vector space with given inner product. Use $\langle\, , \,\rangle_A$ to compute the norm of e_2. ***(3 marks)***

(iii) Explain how to use the Gram-Schmidt procedure on a basis $v_1, \ldots, v_n$ in an inner product space to produce an orthonormal basis. ***(9 marks)***

(iv) Apply Gram-Schmidt to the basis e_2, e_1 (note the order) of $\mathbb{R}^2$ using the inner product $\langle\, , \,\rangle_A$ to get an orthonormal basis. ***(5 marks)***

End of Question Paper