THE NINTH SOMAS UNDERGRADUATE MATHEMATICS TEAM CHALLENGE

Description.

- This is the ninth SoMaS undergraduate mathematics team challenge. The challenge is open to *all* undergraduates. The challenge is a take home team event that consists of a collection of interesting problems suggested by members of staff. These are challenging problems (some much more than others). You should feel proud if you can work out (part of) the solution to any of these problems.

Most importantly.

- This is for fun and you are welcome to tell us anything you like about the problems that you felt were interesting or enjoyable. These observations can include, but are not restricted to, (partial) solutions, special cases, humorous remarks, and generalisations or variations of the problems. Submissions which make the judges smile will be appreciated :-)

To take part.

- Email Fionntan Roukema (f.roukema@sheffield.ac.uk) your solutions before 23:59 on Thursday the 1st of December. Your email should include an (un)imaginative name for your team as well as a list of your team's members (which can be greater than or equal to one).

Other remarks.

- There will be a variety of awards available to people who enter. These will be given for the best team name, the most valuable solution to a problem, the overall best submission, as well as for honourable mentions.

2022-23 SoMaS Challenge Problems

Problem 1: The Secretary of State for Education is hiding in an American style motel with five rooms facing into the car park. The rooms are connected by secret doors which allow the Secretary to move from Room n to Rooms $n \pm 1$ if they exist. You know that:

- if you enter a room and the Secretary is present then they will not run away (it would be undignified to do so);
- if you exit an empty room, the Secretary will scurry though a secret door into an adjacent room;
- the secret doors are so secret that you will never be able to find or use them.

Can you devise a strategy which will allow you to find the Secretary and ask them why your lecturers are on strike?

Problem 2: A fat cat, a middle manager, and an academic would like to share a square cake. Can the fat cat cut the cake into pieces that can be reassembled to form 3 square cakes of different sizes (they need the largest cake, and the academic should be happy with the smallest)? If so, what is the minimal number of pieces that you can find?

Problem 3: Let $a_0, a_1, a_2, \dots$ and $b_0, b_1, b_2, \dots$ be two sequences of real numbers defined by the recurrence relations

$$a_{n+1} = \frac{1}{\sqrt{2}}\sqrt{1 - \sqrt{1 - a_n^2}}, \qquad b_{n+1} = \frac{\sqrt{1 + b_n^2} - 1}{b_n}$$

for $n \geq 0$ and initial terms $a_0 := \dfrac{1}{\sqrt{2}}$, $b_0 := 1$. Show that

$$2^{n+2} a_n \; < \; \pi \; < \; 2^{n+2} b_n.$$

Problem 4: Let $g : \mathbb{R} \to \mathbb{R}$ be a function that is twice differentiable and satisfies

$$g(0) = 2, \quad g'(0) = -2, \quad \text{and } g(1) = 1.$$

Prove that the function

$$f(x) := g(x)g'(x) + g''(x)$$

has at least one root.

Problem 5: What's the largest prime you can find whose first four digits are "2022"? You should explain how you verified that it was prime.

Problem 6: For every football world cup since 1970, Panini have sold collectible sticker albums. A completed album will have a sticker for every squad member at the world cup. This year, there are 670 stickers to collect. Stickers are sold in packets of five. One packet costs 90p, and the five stickers are chosen at random; you won't know which five stickers you have got until you've bought the packet. The album costs £12.99.

(i) What is the expected cost of completing the album?

(ii) Suppose you and a friend buy one album each. You buy sticker packets at the same time, so you can swap duplicates before buying your next packet. What is the expected cost for the person who completes the album first?

Problem 7: SoMaS is going to award an a honorary doctorate to the all-time greatest overall runner. A table of world records is given below. Using data for either women or men:

(i) Plot the data as log (time) versus log (distance);

(ii) Fit a low degree polynomial to the plot in (i);

(iii) Draw some conclusions from the residues and propose a winner for the SoMaS running award.

Table of world records

Sr. No	Distance (m)	Time (sec.) Men	Time (sec.) Women
1	100	9.58	10.49
2	200	19.19	21.34
3	400	43.03	47.6
4	800	100.91	113.28
5	1000	131.96	148.98
6	1500	206	230.07
7	3000	440.67	486.11
8	5000	755.36	846.62
9	10000	1571	1741.03
10	42195	7377	8263

Source: www.worldathlectics.org Outdoor (as of 20/11/2022)