THE FOURTH SOMAS UNDERGRADUATE MATHEMATICS CHALLENGE

Description. This is the fourth SoMaS undergraduate mathematics challenge. The challenge is open to *all* undergraduates. The challenge is a take home team event that consists of 5/6 interesting problems suggested by members of staff. These are challenging problems (some much more than others). You should feel proud if you can work out (part of) the solution to any of these problems.

Above all, this is for fun and you are welcome to tell us anything you like about the problems that you felt were interesting. These observations can include, but are not restricted to, (partial) solutions, special cases, humorous remarks, and generalisations or variations of the problems.

To take part. register a team by emailing:

- James Cranch (J.D.Cranch@shef.ac.uk) or
- Jayanta Manoharmayum (J.Manoharmayum@shef.ac.uk) or
- Jeremy Oakley (j.oakley@sheffield.ac.uk) or
- Fionntan Roukema (f.roukema@shef.ac.uk).

Your email should include an imaginative name for your team as well as a list of your team's members (which can be greater or equal to one).

- You should submit your best solutions by midnight on Friday the 18th of November.

Other remarks.

- With the permission of the relevant team, the best solutions will be posted on the competition website (`http://roukema.staff.shef.ac.uk/somas_challenge.html`).
- There will be a variety of prizes and awards available to people who enter.

2016-2017 Problems.

(1) Consider a two person game in which the most skilled player always wins. Given 32 players of different skill, it is possible to find the best player using a standard knock-out tournament (last 32, last 16, quarterfinals, semifinals, final) in a total of 31 games (16+8+4+2+1).

Given 32 players of different skill, how would you go about finding the second most skilled player and how many games would it take?

(2) Let No8 be the set of positive integers that do not contain the digit 8. For example, $123456790 \in \text{No8}$ but $1234567890 \notin \text{No8}$. Show that

$$\sum_{n \in \text{No8}} \frac{1}{n} \ < \ 80.$$

The bound in the above inequality is not the best possible; what is the best upper bound that you can find for $\sum_{n \in \text{No8}} \frac{1}{n}$?

(3) Monty Hall[1], having run out of cars and goats, has devised a new version of his 'Let's Make a Deal' gameshow. In this game, there are two doors, with a different cash prize behind each. The values of the cash prizes are unknown, other than the fact that one cash prize is twice the value of the other one. You can choose one of the two doors to open, and you get the prize behind that door.

You toss a coin, and based on the coin toss, you choose door 1. You are about to open door 1, when Monty gives you the choice of switching to door 2. He explains that the prize behind door 2 is equally likely to be twice or half the value of the prize behind door 1 (since you tossed a fair coin), so if the prize behind door 1 is X, the expected value of the other prize is

$$0.5 \times 0.5X + 0.5 \times 2X = 1.25X,$$

hence it makes sense to switch. Is his reasoning correct?

(4) Google's basic online calculator says that

$$(\sqrt{5}+2)^{\frac{1}{3}} \ = \ 1.61803398875,$$
$$(\sqrt{5}-2)^{\frac{1}{3}} \ = \ 0.61803398875.$$

Are these two numbers *exactly* distance 1 apart and can you find any/all $(a,b) \in \mathbb{N}^2$ with $(\sqrt{b}+a)^{\frac{1}{3}} - (\sqrt{b}-a)^{\frac{1}{3}} = 1$?

(5) [Guest problem suggested by Dr. Alex Bloemendal of the Broad Institute of Harvard and M.I.T.] (see next page)

[1] if you're not familiar with Monty you should look him up on Google

It is a classical problem to use a biased coin which flips a head with probability $p \in (0,1)$ to produce an experiment with two outcomes of equal probability[2].

Now suppose that you are given a fair coin. For which $p \in [0,1]$ can you use your fair coin to produce an experiment which has one outcome with probability exactly p?

[2]you are encouraged to think of how to do so, or look it up online