Maths Take Home Competition - Elastic Pancakes

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Question 2

Definition 1. A $k \times k$ *border* is $4(k-1)$ pegs arranged to make a square border.

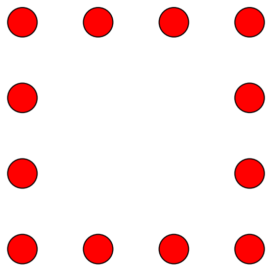

Figure 1: The 4×4 border.

Proposition 2. *Let $k \leq n$. Each square you can make in some $n \times n$ grid A is part of one and only one $k \times k$ border within A.*

Proof. Let S be a formable square on some $n \times n$ grid. If the edges of S are horizontal and vertical, we are done. If not, then form four right angled triangles using each adjacent vertices, forming a $k \times k$ border.

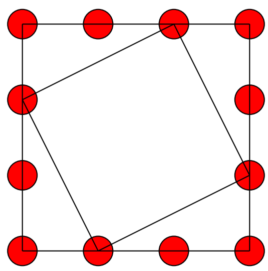

Figure 2: An example of forming a square on a border by using right angled triangles.

It is trivially true that each square is unique to one border. □

Proposition 3. *On an $a \times b$ grid, there are $(a-k+1)(b-k+1)$ $k \times k$ borders, $1 \leq k \leq min(a,b)$.*

Proof. We count the number of $k \times k$ borders in an $a \times b$ grid. Imagine this $k \times k$ border on the most top left hand corner. We can travel $a-k$ right, and $b-k$ down. This gives $(a-k+1)(b-k+1)$ total positions for a $k \times k$ border. □

Proposition 4. *There are $k-1$ squares formable on a $k \times k$ border.*

Proof. We prove that each vertex on a $k \times k$ border belongs to one and only one square on said border. Our trivial case is to choose the four corners of our border, and clearly that is the only square we can make that involves any of those vertices. We note that, if a square is constructable in a $k \times k$ border, using our method for solving Proposition 2, we can make a $k \times k$ square around it. We consider the four right angled triangles that this produces. The sum of the side lengths must sum to $k - 1$, and each way of doing that generates a new square. This generates, (excluding our trivial case) $k - 2$ unique squares, and thus $k - 1$ squares in total.

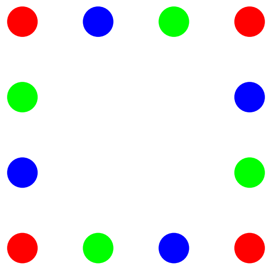

Figure 3: Here are the three squares for the 4×4 case.

□

Theorem 5. *Let $a \leq b$. There are $\frac{1}{12}a(2b-a)(a^2-1)$ squares formable on an $a \times b$ grid.*

Proof. Each $k \times k$ border contains $k - 1$ squares. Each $a \times b$ grid contains $(a - k + 1)(b - k + 1)$ $k \times k$ borders. Therefore the total number of squares on an $a \times b$ grid is

$$\sum_{k=1}^{k=a} (k-1)(a-k+1)(b-k+1)$$

$$= \sum_{k=1}^{k=a} k^3 - (a+b+3)\sum_{k=1}^{k=a} k^2 + (a+b+2a+2b+3)\sum_{k=1}^{k=a} k - (ab+a+b+1)\sum_{k=1}^{k=a} 1$$

$$= \frac{1}{4}a^2(a+1)^2 - \frac{1}{6}a(a+1)(2a+1)(a+b+3) + \frac{1}{2}(a)(a+1)(ab+2a+2b+3) - a(ab+a+b+1)$$

$$= \frac{1}{12}a(2b-a)(a^2-1)$$

as required. □

Corollary 6. *There are $\frac{1}{12}n^2(n^2-1)$ squares formable on an $n \times n$ grid.*

Proof. This is simply Theorem 5, with the case $a = b$. □

We wrote a piece of code entitled `rectangles.py` which calculates the number of rectangles in an $n_1 \times n_2$ grid of pegs (find attached to submission email or see code below). The following table shows the number of rectangles contained inside a rectangle of side lengths given by the row and column of the table (obtained via `rectangles.py`). There are a number of interesting features of this table which we will discuss.

	2	3	4	5	6	7	8	9	10
2	1	3	6	10	15	21	28	36	45
3	3	10	20	33	49	68	90	115	143
4	6	20	44	74	110	152	200	254	314
5	10	33	74	130	198	276	364	462	570
6	15	49	110	198	311	439	586	748	925
7	21	68	152	276	439	632	854	1097	1365
8	28	90	200	364	586	856	1170	1516	1898
9	36	115	254	462	748	1097	1516	1986	2504
10	45	143	314	570	925	1365	1898	2504	3183

1. As we would expect the table is symmetric along the diagonal since there are the same number of rectangles contained in an $a \times b$ rectangle as in a $b \times a$ rectangle.
2. The sequence in the first row is the triangular numbers. That is, there are $\frac{n}{2}(n-1)$ rectangles inside a $2 \times n$ rectangle.
3. The 'first difference' on each row is n^2-2 where n is the number of the row. However, continuing in this vein with second differences, there seems to be no obvious (or pleasant) patterns arising.

The code is:

```
from math import floor

def border_rectangles(n_1,n_2):
    #counter is 1 to account for whole rectangle
    counter = 1
    #consider all possible sizes of gap in bottom right
    for a in range(1,floor(n_2/2)+1):
        #consider all possible heights
        for b in range(1,n_1+1):
            k_1 = (n_1-b)/a
            k_2 = (n_2-a)/b
            #if solution to simultaneous equations
            if k_1==k_2:
                if (k_1.is_integer() or (1/k_1).is_integer()):
                    #symmetric square
                    if a==b and k_1==1:
                        counter += 1
                    #reflection of something else that will be counted
                    elif a==n_2/2:
                        counter+=1
                    #anything else
                    else:
                        counter += 2
    return counter

def rectangles(n_1,n_2):
    #change to number of gaps rather than pegs
    n_1 = n_1-1
    n_2 = n_2-1
```

```
    total = 0
    for i in range(1,n_1+1):
        for j in range(1,n_2+1):
            #count the number of repeats of each border
            number_of_borders = (n_1-i+1)*(n_2-j+1)
            total += number_of_borders*border_rectangles(i,j)
    return total
```

Remark 7. This code was written for Python 3.3. As such, some minor modifications are required for use in Python 2.7.