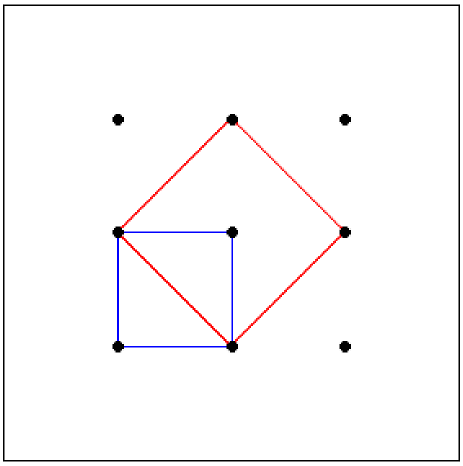
SOMAS MATHS COMPETITION 2015 QUESTON 2 SOLUTION

HOBBIT, HERCULES AND HULK

1. The Question

n^2 pegs are arranged in an $n \times n$ square grid. You have a piece of string which you can wrap around pegs of your choice. How many squares can you form? For example, if $n = 3$ then two such squares are shown below:



2. A Solution

The answer is

$$\frac{n^2(n^2-1)}{12}.$$

Here is a solution. We will demonstrate one way to count the squares for a general $n \times n$ grid and express this as a function of n.

We will define a '**new** $i \times i$ **square**' to be a square that is in an $i \times i$ grid but **not** also in any smaller square grid.

Consider some general $i \times i$ grid like the following :

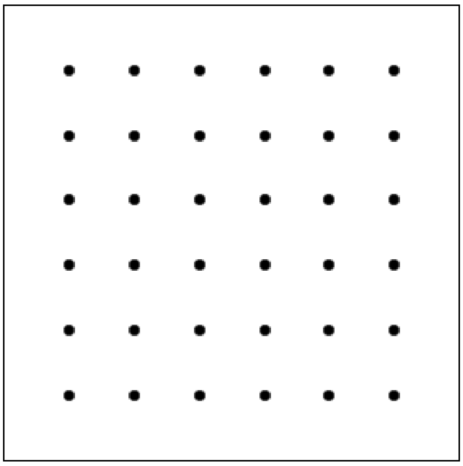


FIGURE 2. A Representation of a blank $i \times i$ grid for $i = 6$.

For a square formed in this grid to be a '**new** $i \times i$ **square**' it is necceassary that the square touches all edges of the perimeter of the $i \times i$ grid.[1] The Figure below aims to demonstrate a one-to-one correspondence between $i-1$ pegs on the edge of the perimeter of the $i \times i$ grid and the number of '**new** $i \times i$ **squares**'.

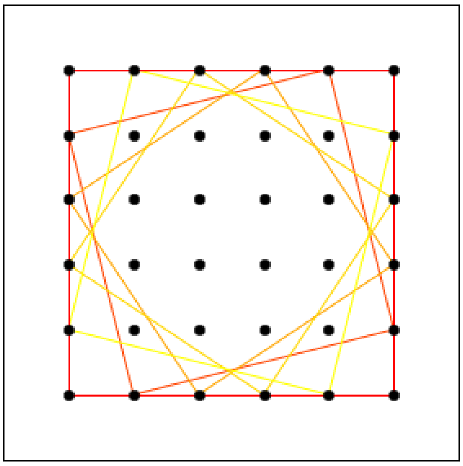


FIGURE 3. A Representation of the one-to-one correspondence between $i-1$ and the new '**new** $i \times i$ **squares**' in an $i \times i$ grid.

So now we know there exists exactly $i-1$ amount of '**new** $i \times i$ **squares**' in an $i \times i$ grid. Hence for any $n \geq i$, for all $n \times n$ grids we can count the number of $i \times i$ grids contained in the $n \times n$ grid and multiply it by the number of '**new** $i \times i$ **squares**'. Since there exist $(n+1-i) \times (n+1-i) = (n+1-i)^2$

[1] Otherwise we would not have a square or a square that can be formed in an $(i-1) \times (i-1)$ grid which would be a contradiction.

amount of unique $i \times i$ grids contained in an $n \times n$ grid and $i-1$ '**new** $i \times i$ **squares**' in each $i \times i$ grids we can now deduce a function for total number of squares we can form in an $n \times n$ grid. Let $S(n)$ be such function, then

$$S(n) = \sum_{i=1}^{i=n} (i-1)(n+1-i)^2 \tag{2.0.1}$$

(2.0.2) by making the substitution $i = j + 1$ we have,

$$= \sum_{j=0}^{j=n-1} (j)(n-j)^2 = \sum_{j=1}^{j=n} (j)(n-j)^2 \tag{2.0.3}$$

$$= \sum_{j=1}^{j=n} j(n^2 - 2nj + j^2) = \sum_{j=1}^{j=n} n^2 j - 2nj^2 + j^3 \tag{2.0.4}$$

$$= n^2 \times \frac{n(n+1)}{2} - 2n \times \frac{n(2n+1)(n+1)}{6} + \frac{n^2(n+1)^2}{4} \tag{2.0.5}$$

$$= \frac{n^2(n+1)}{12}[6n - 4(2n+1) + 3(n+1)] \tag{2.0.6}$$

$$= \frac{n^2(n+1)}{12}(n-1) \tag{2.0.7}$$

$$= \frac{n^2(n^2-1)}{12} \tag{2.0.8}$$

which is exactly what we wanted.

3. Proof by Induction

Here is a formal proof by induction on n to show that the answer is

$$S(n) = \frac{n^2(n^2-1)}{12}.$$

3.1. Base Case.

Consider the case $n = 1$. Then $S(1) = \frac{1^2(1^2-1)}{12} = 0$ as expected.

3.2. Inductive Step.

Let's assume that for some positive natural number k it is true that

$$S(k) = \frac{k^2(k^2-1)}{12}.$$

We will show that $S(k+1) = \frac{(k+1)^2((k+1)^2-1)}{12}$.

First we make the observation that

$$S(k+1) - S(k) \tag{3.2.1}$$

$$= \frac{(k+1)^2((k+1)^2-1)}{12} - \frac{k^2(k^2-1)}{12} \tag{3.2.2}$$

$$= \frac{(k+1)^2(k^2+2k)}{12} - \frac{k^2(k+1)(k-1)}{12} \tag{3.2.3}$$

$$= \frac{k(k+1)^2(k+2)}{12} - \frac{k^2(k+1)(k-1)}{12} \tag{3.2.4}$$

$$= \frac{k(k+1)}{12}[(k+1)(k+2) - k(k-1)] \tag{3.2.5}$$

$$= \frac{k(k+1)}{12}[k^2 + 3k + 2 - k^2 + k] \tag{3.2.6}$$

$$= \frac{k(k+1)}{12}[2(2k+1)] \tag{3.2.7}$$

$$= \frac{k(2k+1)(k+1)}{6}. \tag{3.2.8}$$

So it suffices to show that the number of new squares we can create in a $k+1 \times k+1$ grid that are not also in some fixed $k \times k$ grid within $k+1 \times k+1$ grid the is exactly $\frac{k(2k+1)(k+1)}{6}$.

We can interpret a $k+1 \times k+1$ grid as a $k \times k$ grid with the additional $2k+1$ **red** pegs introduced like so:

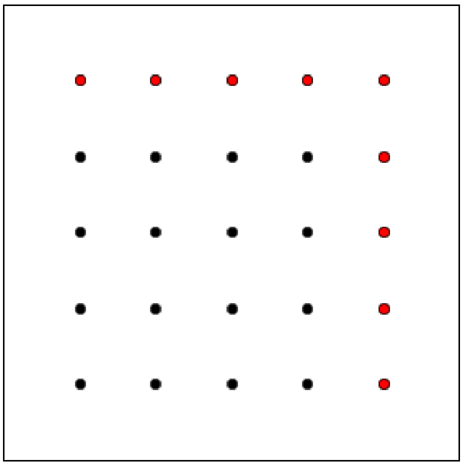


FIGURE 4. How to add $2k+1$ pegs to a $k \times k$ grid to make a $k+1 \times k+1$ grid.

Let's fix some $k \times k$ grid within our $k+1 \times k+1$ grid as shown above. We know that there exist $\frac{k^2(k^2-1)}{12}$ squares using only the black pegs. As we are interested in counting how many new squares are in our $k+1 \times k+1$ grid but not in our $k \times k$ grid we will count how many new squares can be made by using atleast one **red** peg.

We showed before that within an $i \times i$ grid we can make $i-1$ '**new** $i \times i$ **squares**'. But for all $k+1 \geq i$ we can clearly create $2(k+1-i)+1$ $i \times i$ grids that cannot be created within the black grid. This gives a total number of new squares of

(3.2.9) $$\sum_{i=1}^{i=k+1} (i-1)(2(k+1-i)+1)$$

(3.2.10) by making the substitution $i = j+1$ we have,

(3.2.11) $$= \sum_{j=0}^{j=k} j(2(k-j)+1)$$

(3.2.12) $$= \sum_{j=0}^{j=k} 2jk - 2j^2 + j$$

(3.2.13) $$= 2k\frac{k(k+1)}{2} - 2\frac{k(2k+1)(k+1)}{6} + \frac{k(k+1)}{2}$$

(3.2.14) $$= \frac{k(k+1)}{6}[6k - 2(2k+1) + 3]$$

(3.2.15) $$= \frac{k(k+1)}{6}[6k - 4k - 2 + 3]$$

(3.2.16) $$= \frac{k(k+1)(2k+1)}{6}.$$

So we have shown that there are $\frac{k(k+1)(2k+1)}{6}$ new squares formed in a $k+1 \times k+1$ grid that are not present in any fixed $k \times k$ grid within the $k+1 \times k+1$ grid. But we've already shown that

$$\tag{3.2.17}$$
$$S(k+1) - S(k) = \frac{k(k+1)(2k+1)}{6} \tag{3.2.18}$$
$$\implies S(k) + \frac{k(k+1)(2k+1)}{6} = S(k+1) \tag{3.2.19}$$
$$= \frac{(k+1)^2((k+1)^2-1)}{12} \tag{3.2.20}$$

so our induction step is complete.

3.3. **Closing Statement.**

Therefore, for all $n \in \mathbb{N}$ we can form exactly $\frac{n^2(n^2-1)}{12}$ squares in an $n \times n$ grid by joining pegs with string.$\square$

4. Final Thoughts and Questions

(1) Given that we now have a formula for the answer, how else can we interpret this?

For example $S(n) = \frac{n^2(n^2-1)}{12} = \frac{n^2!}{(n^2-2)!2!} \times \frac{1}{6} = \binom{n^2}{2}\frac{1}{6}$.

We can interpret this as the number of ways of choosing 2 pegs in our $n \times n$ grid and then dividing by 6. Does this hint at some relationship with counting pegs on the face of a $n \times n \times n$ cube and then dividing by 6 due to the symetries of the cube?

Similarly, another interpretation is given by $S(n) = \frac{n^2(n^2-1)}{12} = \frac{(n+1)!}{(n-2)!3!}\frac{n}{2} = \binom{n+1}{3}\frac{n}{2}$.

What other interpretations exist?

(2) The Proof by Induction argument showed us that we can that $S(k+1) - S(k) = \frac{k(k+1)(2k+1)}{6} = \sum_{i=1}^{i=k} i^2$. This statement implies that

The number of squares that can be formed on a $k \times k$ grid by joining dots with straight lines is equal to the number of squares that can be formed on a $k+1 \times k+1$ grid only using diagonal lines.

Is there an intuitive reason why?

(3) We can now deduce that the number of squares that can be formed using *non-vertical* and *non-horizontal* lines in an $n \times n$ grid is

$$\frac{n^2(n^2-1)}{12} - \frac{n(2n-1)(n-1)}{6} \tag{4.0.1}$$

$$= \frac{n(n+1)}{12}[n(n-1) - 2(2n-1)] \tag{4.0.2}$$

$$= \frac{n(n+1)}{12}[n^2 - n - 4n + 2] \tag{4.0.3}$$

$$= \frac{n(n-1)^2(n-2)}{12}. \tag{4.0.4}$$

From this we can deduce that the proportion of squares that are formed using *non-vertical* and *non-horizontal* lines in an $n \times n$ grid is given by

$$\frac{\frac{n(n-1)^2(n-2)}{12}}{\frac{n^2(n^2-1)}{12}} = \frac{(n-1)^2(n+2)}{n(n+1)(n-1)} \tag{4.0.5}$$

$$= \frac{(n-1)(n-2)}{n(n+1)} \tag{4.0.6}$$

which tends to 1 as $n \to \infty$.

(4) There seems to be lots of geometric interpretations we can make using the different formulae that were produced. Could we use these to create a simple 'proof from the book'?

(5) How will formulas vary for different polygons, lattices and dimensions?

For example, what does a two variable function $S_D : \mathbb{N}^2 \to \mathbb{N}$ such that $S_D(n)$ returns the number of 'D dimensional cubes' in an n^D lattice look like[2]?

If we suppose that $S_D(n)$ is always a finite polynomial then if we can always exhaustively check the value of $S_D(n)$ for arbitarily large n then we should be able to obtain the co-efficients of the polynomial by solving simultaneous equations in matrix form.

(6) What about the question '*In an $n \times n \times n$ cubic grid, how many cubes can be formed by connecting pegs with string?*'

We can consider 3 classes of cubes that we can create within our $n \times n \times n$ cubic grid:

(a) the cube has 6 faces that are parallel to some face of the $n \times n \times n$ cube,

(b) the cube has 2 faces that are parallel to some face of the $n \times n \times n$ cube,

(c) the cube has 0 faces that are parallel to some face of the $n \times n \times n$ cube.

[2] If it exists.

We know there are $\frac{n^2(n+1)^2}{4}$ cubes of class (a).

For a cube to be of class (b) it must have one square face formed of *non-vertical* and *non-horizontal* on an $n \times n$ grid. Let's suppose we have such a square face in the x, y plane. We can call the displacent of each line of our square face in the x and y axis X and Y respectively. Then in order to for this square to form a cube in our $n \times n \times n$ cubic grid it's vertical displacement in the z axis,Z, given by $Z = \sqrt{X^2 + Y^2}$ must be an integer. So X, Y, Z must correspond to a pythagorean triple.

I have not yet discovered any cubes of class (c).[3] Here is an interesting 3D structure formed whilst trying to create class (c) cubes

`https://www.youtube.com/watch?v=IXKJf1wRIW8` .

Does a cubic analogue exist and, if so, would it tessalate in 3D space?

[3] Although I suspect they do exist.