**MAS277 2011-12 Solution 1**

(i) and (ii)(a) are bookwork. (ii)(b,c) are unseen. (b) is only there to help with (c).

(i) If $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$, then $\lambda_1 = \cdots = \lambda_n = 0$.

Every vector $v \in V$ can be expressed as $\lambda_1 v_1 + \cdots + \lambda_n v_n$, for some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

The vectors must both be linearly independent and span $V$.

(ii)(a) Firstly, we have $\sin' = \cos$ and $\cos' = -\sin$, so $\sin'' = -\sin$ and $\cos'' = -\cos$, so $\sin \in V$ and $\cos \in V$. Hence $g$, being a linear combination of $f$, sin and cos, is in $V$.

$$\begin{aligned} g(0) &= f(0) - a\sin(0) - b\cos(0) = b - a.0 - b.1 = 0 \\ g'(0) &= f'(0) - a\sin'(0) - b\cos'(0) = a - a\cos(0) + b\sin(0) = a - a.1 - b.0 = 0. \end{aligned}$$

Hence $h(0) = 0$.

As $g'' = -g$, we have

$$h'(x) = 2g(x)g'(x) + 2g'(x)g''(x) = 2g'(x)(g(x) + g''(x)) = 0.$$

This means that $h$ is constant, but $h(0) = 0$, so $h(x) = 0$ for all $x$.

However, $h(x) = g(x)^2 + g'(x)^2$, which is the sum of two nonnegative quantities; the only way we can have $h(x) = 0$ is if $g(x) = 0 = g'(x)$.

Hence $f(x) - a\sin(x) - b\cos(x) = 0$, so $f = a\sin + b\cos$, and therefore sin and cos span $V$.

(b) If $f \in V$ then $f + f'' = 0$. Differentiating, $f' + f''' = 0$. Thus $V \subseteq W$.

If $f \in W$ then $f' + (f')'' = 0$ so $f' \in U$.

(c) Let $\lambda, \mu, \rho \in \mathbb{R}$ be such that

$$\lambda j + \mu\sin + \rho\cos = 0.$$

Then $0 = \lambda j(0) + \mu\sin(0) + \rho\cos(0) = \lambda + \rho$ so $\rho = -\lambda$. Also $0 = \lambda j(\pi) + \mu\sin(\pi) + \rho\cos(\pi) = \lambda - \rho$ so $\rho = \lambda$. Therefore $\lambda = \rho = 0$ and, as then $0 = \mu\sin(\pi/2) = \mu$, $\mu = 0$ also. Thus $j, \sin, \cos$ are linearly independent.

Note that $j \in W$, as $j' = j''' = 0$ and $\sin, \cos \in W$ by (b).

Now let $b \in W$. Then $b' \in V$ so, by (a), $b' = \alpha\sin + \beta\cos$ for some $\alpha, \beta \in \mathbb{R}$. Hence $b = -\alpha\cos + \beta\sin + \gamma j$ for some $\gamma \in \mathbb{R}$. Thus $j, \sin, \cos$ span $W$ and, as they are also linearly independent, they form a basis for $W$.

**MAS277 2011-12 Solution 2**

(i) and (ii) are bookwork. (iii) and (iv) are unseen but similar to seen.

(i) As $U$ is a subspace we have $0_V \in U$, and as $W$ is a subspace we have $0_V \in W$, so $0_V \in U \cap W$.

Next, suppose we have $v, v' \in U \cap W$, $\alpha, \alpha' \in \mathbb{R}$.

Then $\alpha v + \alpha' v' \in U$, as $U$ is a subspace, and $\alpha v + \alpha' v' \in W$, as $W$ is a subspace. Therefore $\alpha v + \alpha' v' \in U \cap W$.

So $U \cap W$ is a subspace.

$U + W = \{v \in V \mid v = u + w \text{ for some } u \in U,\ w \in W\}$.

Let $u \in U$ and $w \in W$. Then $u = u + 0_V \in U + W$ and $w = 0_V + w \in U + W$. Thus $U \subseteq U + W$ and $W \subseteq U + W$

(ii)(a) $\dim(U) \leq \dim(V)$.

(b) $U = V$.

(c) $\dim(U) + \dim(W) = \dim(U \cap W) + \dim(U + W)$.

(iii)(a) By (ii)(a), $3 = \dim(V) \geq \dim(U + W) \geq \dim U = 2$ so $\dim(U + W) = 2$ or 3.

Suppose $\dim(U + W) = 2$. By (ii)(b), $U = U + W$ so $W \subseteq U$.

But then, by (ii)(b), $W = U$ which is false.

Therefore $\dim(U + W) = 3 = \dim(V)$ and, by (ii)(b), $U + W = V$.

(b) By (ii)(c),

$$\dim(U \cap W) = \dim(U) + \dim(W) - \dim(U + W) = 2 + 2 - 3 = 1.$$

(iv)(a)

$$\begin{aligned} V &= \{(a, b, c, -a - b - c)^T | a, b, c \in \mathbb{R}\}, \\ U &= \{(a, -a, c, -c)^T | a, c \in \mathbb{R}\} \text{ and} \\ W &= \{(a, b, a, -b - 2a)^T | a, b \in \mathbb{R}\}. \end{aligned}$$

$\dim(V) = 3$, $\dim(U) = 2 = \dim(W)$.

By (iii)(b), $\dim(U \cap W) = 1$ so any non-zero element of $U \cap W$ spans $U \cap W$. Take $v = (1, -1, 1, -1)^T$.

Let $u' = u + v = (3, -3, 8, -8)^T \in U$ and $w' = w - v = (2, -4, 2, 0)^T \in W$. Then $u' + w' = u + v + w - v = U = W$ but $u' \neq u$.

# MAS277 2011-12 Solution 3

(i) is bookwork. The rest is fairly standard though linear maps of this particular type are unseen.

(i) The kernel is the set $\{v \in V \mid \phi(v) = 0\}$.
The image is the set $\{w \in W \mid w = \phi(v) \text{ for some } v \in V\}$.

$$\dim(\ker(\phi)) + \dim(\operatorname{im}(\phi)) = \dim(V).$$

(ii)(a)

$$\begin{aligned}
\phi(E_1) &= \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} = -E_2 + 2E_3. \\
\phi(E_2) &= \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 0 & 2 \end{pmatrix} = -2E_1 - 3E_2 + 2E_4. \\
\phi(E_3) &= \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix} = E_1 + 3E_3 - E_4. \\
\phi(E_4) &= \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} = E_2 - 2E_3.
\end{aligned}$$

The matrix of $\phi$ with respect to the given basis is therefore

$$\begin{pmatrix} 0 & -2 & 1 & 0 \\ -1 & -3 & 0 & 1 \\ 2 & 0 & 3 & -2 \\ 0 & 2 & -1 & 0 \end{pmatrix}.$$

$\phi(I_2) = A - A = 0$ so $I_2 \in \ker(\phi)$. $\phi(A) = A^2 - A^2 = 0$ so $A \in \ker(\phi)$. $I_2$ and $A$ are clearly linearly independent so $\dim(\ker(\phi)) \geq 2$.

By (i), $\dim(\operatorname{im}(\phi)) = 4 - \dim(\ker(\phi)) \leq 2$.

$\phi(E_1) = -E_2 + 2E_3$ and $\phi(E_2) = -2E_1 - 3E_2 + 2E_4$ are clearly linearly independent.

Therefore $\dim(\operatorname{im}(\phi)) \geq 2$. Hence $\dim(\operatorname{im}(\phi)) = 2$ and $\dim(\ker(\phi)) = 4 - \dim(\operatorname{im}(\phi)) = 2$.

(c) For $1 \leq i \leq 4$, $\phi(E_i)$ has trace 0 so $\phi(E_i) \in W$.

These elements span $\operatorname{im}(\phi)$ therefore $\operatorname{im}(\phi) \subseteq W$.

$\operatorname{im}(\phi) \neq W$ because $\dim(W) = 3 \neq \dim(\operatorname{im}(\phi))$.

(d) $f(A^2) = A^3 - A^3 = 0$ so $A^2 \in \ker \phi$.

$$A^2 = \begin{pmatrix} 2 & 3 \\ 6 & 11 \end{pmatrix} = 2I_2 + 3A.$$

**MAS277 2011-12 Solution 4**

(i,ii) are bookwork, (iii) is unseen standard.

An *inner product* on $V$ is a rule that gives a number $\langle u, v\rangle \in \mathbb{R}$ for each $u, v \in V$, with the following properties:

(a) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$.

(b) $\langle \alpha u, v\rangle = \alpha\langle u, v\rangle$ for all $u, v \in V$ and $\alpha \in \mathbb{R}$.

(c) $\langle u, v\rangle = \langle v, u\rangle$ for all $u, v \in V$.

(d) We have $\langle u, u\rangle \geq 0$ for all $u \in V$, and $\langle u, u\rangle = 0$ if and only if $u = 0$.

For $v, w \in V$, $|\langle v, w\rangle| \leq \|v\|\,\|w\|$, with equality if and only if $v$ and $w$ are linearly dependent.

Put $g(x) = 1 - x^3$. Then

$$\|g\|^2 = \langle g, g\rangle = \int_0^1 (1 - x^3)^2 dx = \int_0^1 1 - 2x^3 + x^6, dx = [x - x^4/2 + x^7/7]_0^1 = 9/14.$$

Also, $\|f\|^2 = \int_0^1 f(x)^2\,dx$, and $\langle f, g\rangle = \int_0^1 (1 - x^3) f(x)\,dx$.

By the Cauchy-Schwarz inequality,

$$\left|\int_0^1 (1 - x^3) f(x)\,dx\right| \leq \frac{3}{\sqrt{14}} \sqrt{\int_0^1 f(x)^2\,dx}.$$

Put $v_1 = 1$. Then $\langle v_1, v_1\rangle = \int_0^1 1\,dx = 1$.

Next, put $u_2 = x$, and try to find $\lambda$ such that $v_2 = u_2 - \lambda v_1$ is orthogonal to $v_1$, i.e., $\lambda = \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle}$. As $\langle u_2, v_1\rangle = \int_0^1 x\,dx = [x^2/2]_0^1 = 1/2$, we have $\lambda = 1/2$, and $v_2 = x - 1/2$.

Finally, put $u_3 = x^4$, and try to find $\lambda$ and $\mu$ such that $v_3 = u_3 - \lambda v_1 - \mu v_2$ is orthogonal to $v_1$ and $v_2$. Then $\lambda = \frac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle}$ and $\mu = \frac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle}$.

But

$$\langle u_3, v_1\rangle = \int_0^1 x^4\,dx = [x^5/5]_0^1 = 1/5,$$

$$\langle u_3, v_2\rangle = \int_0^1 (x^5 - x^4/2)\,dx = [x^6/6 - x^5/10]_0^1 = 1/15$$

and

$$\langle v_2, v_2\rangle = \int_0^1 (x^2 - x + 1/4)\,dx = [x^3/3 - x^2/2 + x/4]_0^1 = 1/12,$$

so

$$v_3 = x^4 - 1/5 - 4(x - 1/2)/5 = x^4 - 4x/5 + 1/5.$$

An orthogonal basis is $\{1, x - 1/2, x^4 + (1 - 4x)/5\}$.

**MAS277 2011-12 Solution 5**

(i) is bookwork, (ii,iii) are unseen standard.

(i) A set $\mathcal{V} = \{v_1, \ldots, v_n\}$ of elements of $V$ is *orthogonal* if we have $\langle v_i, v_j \rangle = 0$ for all $i \neq j$ and is *strictly orthogonal* if it is orthogonal, and all the elements $v_i$ are nonzero.

Let $\mathcal{V} = \{v_1, \ldots, v_n\}$ be a strictly orthogonal set, and suppose we have a linear relation $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$. Let $1 \leq i \leq n$. Then

$$\langle v_i, \lambda_1 v_1 + \cdots + \lambda_n v_n \rangle = \langle v_i, 0 \rangle = 0.$$

The left hand side is

$$\lambda_1 \langle v_i, v_1 \rangle + \lambda_2 \langle v_i, v_2 \rangle + \cdots + \lambda_n \langle v_i, v_n \rangle.$$

Moreover, the inner products $\langle v_i, v_j \rangle$ are zero unless $j = i$, so the only nonzero term on the left hand side is $\lambda_i \langle v_i, v_i \rangle$. We conclude that $\lambda_i \langle v_i, v_i \rangle = 0$.

The set is strictly orthogonal, so $v_i \neq 0$, so $\langle v_i, v_i \rangle > 0$. It follows that $\lambda_i = 0$ for all $i$. Thus $\mathcal{V}$ is linearly independent.

(ii) We have $\sin t \cos 5t = (\sin 6t - \sin 4t)/2$ and $\sin 2t \cos 4t = (\sin 6t - \sin 2t)/2$.

Then

$$\langle \sin t \cos 5t, \sin 2t \cos 4t \rangle = \langle \sin 6t, \sin 6t \rangle / 4 = \pi/4,$$

and

$$\langle \sin t \cos 5t, \sin t \cos 5t \rangle = \langle \sin 2t \cos 4t, \sin 2t \cos 4t \rangle = \pi/2,$$

so that the cosine of the angle between the two functions is $\frac{\pi/4}{\sqrt{\pi/2.\pi/2}} = 1/2$, and the angle is $\pi/3$.

The distance between $f$ and $g$ is

$$\sqrt{\langle f-g, f-g \rangle} = \sqrt{(\langle f, f \rangle - 2\langle f, g \rangle + \langle g, g \rangle)} = \sqrt{(\pi/2 - \pi/2 + \pi/2)} = \sqrt{\pi/2}.$$

(iii) We have

$$\langle \phi(v), A \rangle = \text{trace}\left( \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} \beta & \alpha \\ 0 & \gamma \end{pmatrix} \right) = a\beta + b\alpha + d\gamma.$$

We want to define $\hat{\phi}(A) = p + q\cos t + r \sin t$ so that $\langle \hat{\phi}(A), v \rangle$ agrees with the value above.

But

$$\langle p + q\cos t + r \sin t, \alpha + \beta \cos t + \gamma \sin t \rangle = 2\pi p\alpha + \pi q\beta + \pi r\gamma,$$

so we need $p = \frac{b}{2\pi}$, $q = \frac{a}{\pi}$, $r = \frac{d}{\pi}$. Thus

$$\hat{\phi}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{b + 2a\cos t + 2d \sin t}{2\pi}.$$